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ABSTRACT

This unit is 1 of 12 developed for the university classroom portion of the Mathematics-Methods Program (MMP), created by the Indiana University Mathematics Education Development Center (MEDC) as an innovative program for the mathematics training of prospective elementary school teachers (PSTs). Each unit is written in an activity format that involves the PST in doing mathematics with an eye toward application of that mathematics in the elementary school. This document is one of four units that are devoted to the basic number work in the elementary school. In addition to an introduction to the unit, the text has sections on the conceptual development of multiplication and division, developing the basic multiplication and division facts, and computational algorithms for multiplication and division. (MP)

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PREFACE

The Mathematics-Methods Program (MMP) has been developed by the Indiana University Mathematics Education Development Center (MEDC) during the years 1971-75. The development of the MMP was funded by the UPSTEP program of the National Science Foundation, with the goal of producing an innovative program for the mathematics training of prospective elementary school teachers (PSTs).

The primary features of the MMP are:

- It combines the mathematics training and the methods training of PSTs.
- It promotes a hands-on, laboratory approach to teaching in which PSTs learn mathematics and methods by doing rather than by listening, taking notes or memorizing.
- It involves the PST in using techniques and materials that are appropriate for use with children.
- It focuses on the real-world mathematical concerns of children and the real-world mathematical and pedagogical concerns of PSTs.

The MMP, as developed at the MEDC, involves a university classroom component and a related public school teaching component. The university classroom component combines the mathematics content courses and methods courses normally taken by PSTs, while the public school teaching component provides the PST with a chance to gain experience with children and insight into their mathematical thinking.

A model has been developed for the implementation of the public school teaching component of the MMP. Materials have been developed for the university classroom portion of the MMP. These include 12 instructional units with the following titles:

Numeration
Addition and Subtraction
Multiplication and Division
Rational Numbers with Integers and Reals
Awareness Geometry
Transformational Geometry
Analysis of Shapes
Measurement
Number Theory
Probability and Statistics
Graphs: the Picturing of Information
Experiences in Problem Solving

These units are written in an activity format that involves the PST in doing mathematics with an eye toward the application of that mathematics in the elementary school. The units are almost entirely independent of one another, and any selection of them can be done, in any order. It is worth noting that the first four units listed pertain to the basic number work in the elementary school; the second four to the geometry of the elementary school; and the final four to mathematical topics for the elementary teacher.

For purposes of formative evaluation and dissemination, the MMP has been field-tested at over 40 colleges and universities. The field implementation formats have varied widely. They include the following:

- Use in mathematics department as the mathematics content program, or as a portion of that program;
- Use in the education school as the methods program, or as a portion of that program,
- Combined mathematics content and methods program taught in

either the mathematics department, or the education school, or jointly;

- Any of the above, with or without the public school teaching experience.

Common to most of the field implementations was a small-group format for the university classroom experience and an emphasis on the use of concrete materials. The various centers that have implemented all or part of the MMP have made a number of suggestions for change, many of which are reflected in the final form of the program. It is fair to say that there has been a general feeling of satisfaction with, and enthusiasm for, MMP from those who have been involved in field-testing.

A list of the field-test centers of the MMP is as follows:

ALVIN JUNIOR COLLEGE
Alvin, Texas

BLUE MOUNTAIN COMMUNITY COLLEGE
Pendleton, Oregon

BOISE STATE UNIVERSITY
Boise, Idaho

BRIDGEWATER COLLEGE
Bridgewater, Virginia

CALIFORNIA STATE UNIVERSITY, CHICO

CALIFORNIA STATE UNIVERSITY, NORTHRIDGE

CLARKE COLLEGE
Dubuque, Iowa

UNIVERSITY OF COLORADO
Boulder, Colorado

UNIVERSITY OF COLORADO AT DENVER

CONCORDIA TEACHERS COLLEGE
River Forest, Illinois

GRAMBLING STATE UNIVERSITY
Grambling, Louisiana

ILLINOIS STATE UNIVERSITY
Normal, Illinois

INDIANA STATE UNIVERSITY EVANSVILLE

INDIANA STATE UNIVERSITY
Terre Haute, Indiana

INDIANA UNIVERSITY
Bloomington, Indiana

INDIANA UNIVERSITY NORTHWEST
Gary, Indiana

MACALESTER COLLEGE
St. Paul, Minnesota

UNIVERSITY OF MAINE AT FARMINGTON

UNIVERSITY OF MAINE AT PORTLAND-GORHAM

THE UNIVERSITY OF MANITOBA
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UNIVERSITY OF NORTHERN IOWA
Cedar Falls, Iowa

NORTHERN MICHIGAN UNIVERSITY
Marquette, Michigan

NORTHWEST MISSOURI STATE UNIVERSITY
Maryville, Missouri

NORTHWESTERN UNIVERSITY
Evanston, Illinois

OAKLAND CITY COLLEGE
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UNIVERSITY OF OREGON
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SAN DIEGO STATE UNIVERSITY
San Diego, California

SAN FRANCISCO STATE UNIVERSITY
San Francisco, California

SHELBY STATE COMMUNITY COLLEGE
Memphis, Tennessee

UNIVERSITY OF SOUTHERN MISSISSIPPI
Hattiesburg, Mississippi

SYRACUSE UNIVERSITY
Syracuse, New York

TEXAS SOUTHERN UNIVERSITY
Houston, Texas

WALTERS STATE COMMUNITY COLLEGE
Morristown, Tennessee

WARTBURG COLLEGE
Waverly, Iowa

WESTERN MICHIGAN UNIVERSITY
Kalamazoo, Michigan

WHITTIER COLLEGE
Whittier, California

UNIVERSITY OF WISCONSIN--RIVER FALLS

UNIVERSITY OF WISCONSIN/STEVENS POINT

THE UNIVERSITY OF WYOMING
Laramie, Wyoming

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CONTENTS

INTRODUCTION TO THE MULTIPLICATION AND DIVISION UNIT

This unit focuses on developing techniques for teaching multiplication and division of whole numbers. Just as subtraction presents more difficulty for children than addition, you will find that division usually presents more difficulty than multiplication.

Multiplication and division arise quite naturally from the child's real world. For this reason the unit begins by developing an understanding of the models that can be used to interpret multiplication and division situations in the real world. Section I focuses on introducing the concepts of multiplication and division, Section II develops techniques for helping children learn basic multiplication and division facts, and Section III develops the computational algorithms for multiplication and division.

There is a great stress in this unit on developing the real-world problem ⟶ model ⟶ symbol instructional strategy. This progression from a problem to symbols is an important part of a point of view toward planning the mathematical instruction of children. Another emphasis in this unit is the importance of helping children develop thinking patterns and strategies. These thinking patterns are important from the immediate, pragmatic point of view of helping children learn the basic facts; and they are also important from the long-range point of view of learning mathematics in general, since they can help develop children's confidence in their own ability to think and reason numerically. Finally, important mathematical and pedagogical aspects of the basic number properties are developed in this unit. Throughout the unit the prospective teacher is challenged with varied puzzles and problems.

There are times when a teacher needs to call upon his or her resources to motivate, reinforce, or provide enrichment opportunities for children. Games, wisely selected and used, can be quite useful for these purposes. You are encouraged to keep a game file and to add to it from your unit experiences and personal study.

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Section I

THE CONCEPTUAL DEVELOPMENT OF MULTIPLICATION AND DIVISION FACTS

This section is concerned with issues and considerations related to introducing the operations of multiplication and division to young children. One area of emphasis in this section, as in the entire unit, is the problem → model → symbol instructional sequence. This sequence is described in the Overview, which is part of Activity 1, and in Activity 2. Activity 1 also identifies three stages in the child's learning of multiplication and division: the conceptual development of multiplication and division; the learning of basic facts; and the learning of the algorithms for multiplication and division. The remaining activities of Section I focus on the conceptual development of multiplication and division. These activities provide an opportunity to study a sequence of introductory topics (Activity 3), to discuss children's thinking related to solving multiplication and division examples (Activity 4), and to examine special problems related to division with remainders (Activity 5).

This section provides numerous opportunities for you to consider examples of techniques which are appropriate for teaching young children. The practice provided should develop your confidence in your own ability to teach multiplication and division concepts to young children. As you work through the activities be sure to record questions and concerns so that you can bring them to your instructor's attention in the seminar which concludes this section.

MAJOR QUESTIONS

1. Describe how you would develop the concept of multiplication (or division) using a real-world problem ⟶ model ⟶ symbol strategy. Use an example as you describe your strategy.

2. "Mr. Jones has 32 desks to arrange in the classroom. If he wants six rows, how many desks should be in each row?" Discuss why this problem might present special difficulties for young children. What considerations would you keep in mind when teaching the class of problems to which this belongs?

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ACTIVITY 1

AN OVERVIEW OF MULTIPLICATION AND DIVISION IN THE ELEMENTARY SCHOOL

FOCUS:

This activity will present a general overview of the scope and sequence of multiplication and division in the elementary school. Three stages of development are identified and an instructional point of view is described.

MATERIALS:

Some elementary mathematics textbooks (grades 2, 3, and 4); the Mathematics Methods Program slide-tape presentation, "Multiplication and Division in the Elementary School" (optional).

DIRECTIONS:

1. Read and discuss the essay "An Overview of Multiplication and Division in the Elementary School" (or view and discuss the slide-tape presentation).
2. Using a textbook choose any two examples (one multiplication example and one division) and illustrate a model for each example.
3. Use the textbooks to help you fill in an appropriate sequence of multiplication and division examples, ranging from easy to hard.

<u>Multiplication</u> - $\boxed{\text{Easy}}$ to $\boxed{\text{Hard}}$

$$\begin{array}{r} 24 \\ \underline{\times \ \ 8} \end{array} \qquad \qquad \begin{array}{r} 5784 \\ \underline{\times \ 437} \end{array}$$

<u>Division</u> - $\boxed{\text{Easy}}$ to $\boxed{\text{Hard}}$

$$6\overline{)132} \qquad \qquad 428\overline{)96843}$$

4. Which do you feel is more difficult for children, multiplication or division? Why? Is there any analogy to addition and subtraction?

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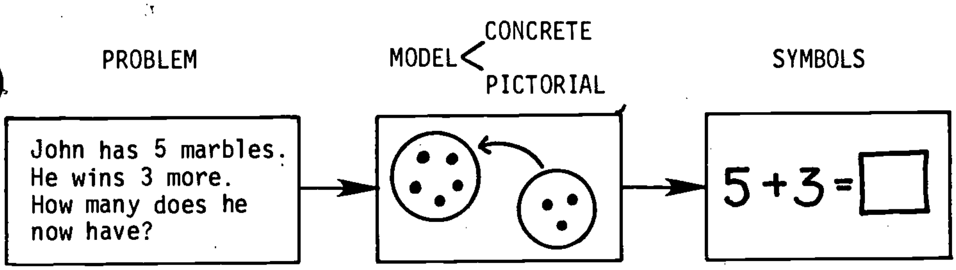
AN OVERVIEW OF MULTIPLICATION AND DIVISION IN THE ELEMENTARY SCHOOL

Throughout the history of teaching mathematics, there has been a consistent belief that children should be introduced to mathematical concepts through real-world examples. In the early grades, addition and subtraction are introduced using the joining and separating of concrete and pictured objects. This is done as part of a sequence which begins by posing some problem, then selecting a concrete or pictorial model to represent the problem, and finally, representing the problem with mathematical symbols. This problem-to-symbol progression provides a sound basis for all mathematics instruction for children.



So in working with multiplication and division, whether developing concepts, basic facts, or algorithms, the teacher should start with real-world problems, work with concrete or pictorial models of problems, and only then introduce mathematical symbolism.

This overview will emphasize the application of the problem-to-symbol progression to the stages in learning about multiplication and division. Before reading on you should be aware of two other considerations. One is that it is often important to deal with the progression in reverse order, i.e., to give children experience creating real-world problems to fit a particular model. The other consideration is that at each stage of the learning of multiplication and di-

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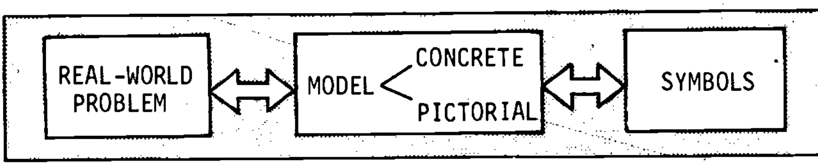
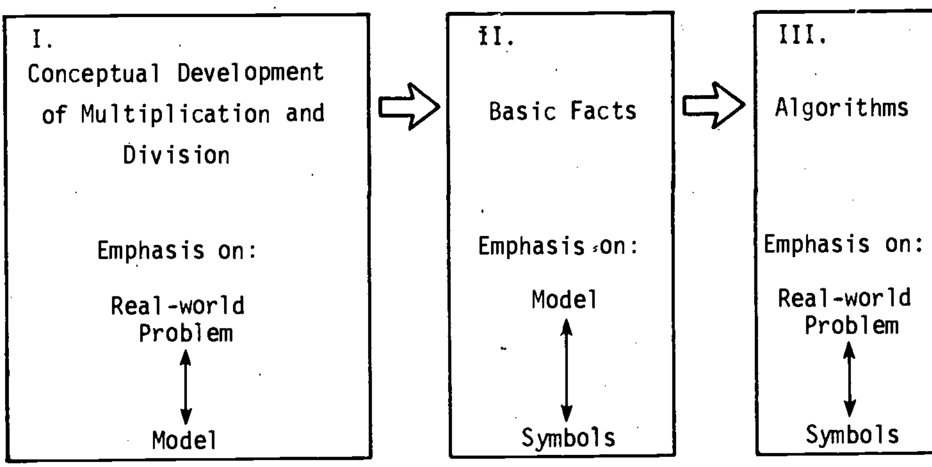
vision a different part of the progression is emphasized; e.g., when teaching basic facts one emphasizes the model⟷symbol part.

The following diagrams represent the two major ideas discussed in the overview and the interaction between these ideas.

INSTRUCTIONAL POINT OF VIEW



STAGES IN LEARNING MULTIPLICATION AND DIVISION



The order in which children usually learn about multiplication and division is essentially parallel to their learning of addition and subtraction--except that it occurs about two to three years later in their school experience. The order generally proceeds as follows:

I. The concept of multiplication is developed, followed by a development of division.

II. The basic number facts for multiplication and division are introduced and memorized.

III. The algorithms for multiplication and division are developed in an easy-to-hard sequence.

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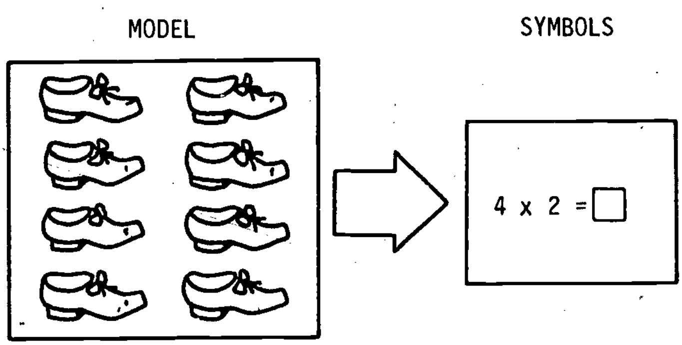
Each of these three developmental stages is described in more detail below. The references to grade levels are approximate and are provided in an attempt to give you some insight into the age of the children concerned at each stage.

STAGE I: CONCEPTUAL DEVELOPMENT

The conceptual development of multiplication usually begins at the end of grade 2. This development is often introduced through stories such as,

> "Mike has four pairs of shoes (or four nickels, etc.).
> How many shoes (cents) does he have in all?"

Pictures or concrete materials are used to represent the story and, finally, symbolism is introduced.

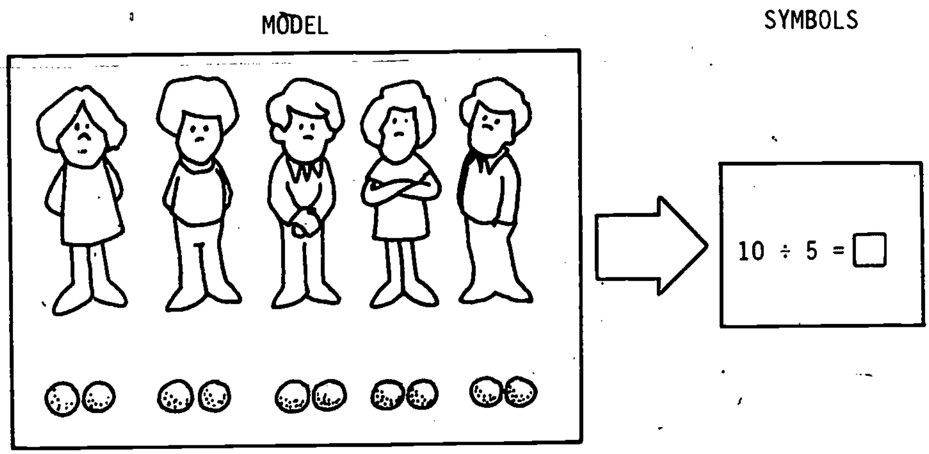


Children at this level often find the answer by repeated addition (2 + 2 + 2 + 2) or by skip-counting by 2's. There is no attempt at this stage of the development to formalize the manipulation of symbols or to memorize basic number facts.

Like multiplication, division is introduced at the end of grade 2, through stories such as:

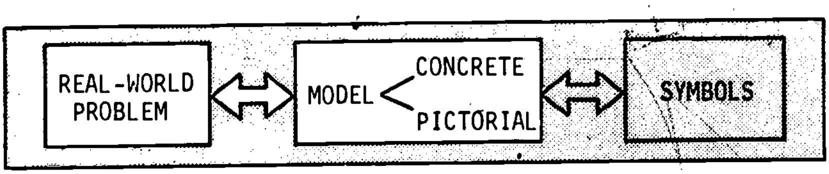
> "There are 10 cookies to be shared among five children.
> How many cookies should each child get?"

Such a story is often acted out in the classroom with five children standing while a plate of 10 cookies is distributed.



Symbols are eventually used to represent this situation. As with multiplication, children are not pressed to formalize their processing of the numbers. In fact, they should be encouraged to develop their own strategies for finding the answer. Some children will think, "One for you, one for you, one for you, etc.," going up and down the line until all the cookies are gone, and then look back to see how many each child has. Another child might hand out three to each child and discover that there are not enough. Similarly a child might pass out two cookies each and find that the cookies had been shared evenly!

The emphasis in the conceptual development should be on relating the real world and the model for multiplication and division.



It is important that children learn which problems should be solved by multiplication and division. The use of concrete or pictorial aids to represent the real-world situation is essential for

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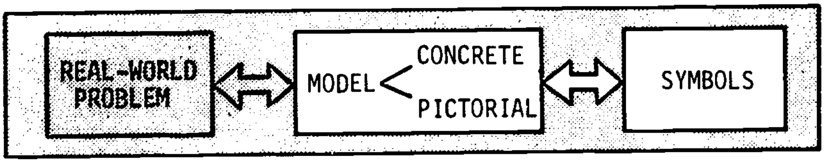
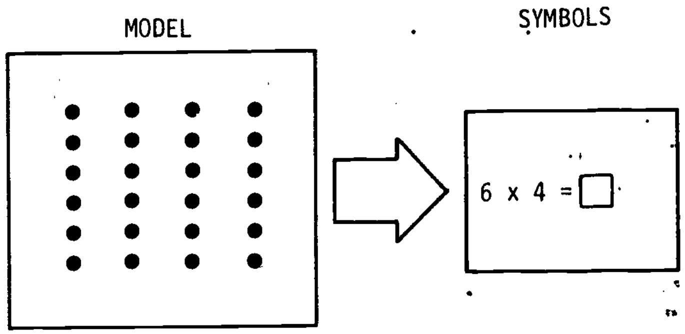
the conceptual development. The interpretation of a problem using a model represents a semi-abstraction of the problem situation and helps the child bridge the gap between the problem and the mathematical symbols used to represent that problem.

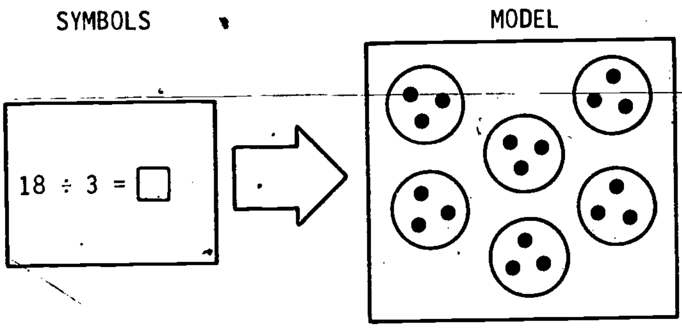
STAGE II: THE BASIC FACTS

After a careful introduction to and development of the concepts of multiplication and division, the children are encouraged to learn the basic number facts. The basic facts refer to combinations in which the factors are less than 10. Thus, 6 x 8 is a basic fact, while 22 x 8 is not. Division basic facts are those combinations related to the multiplication basic facts. Thus, 63 ÷ 7 is a basic fact, while 132 ÷ 6 is not. At this stage, the emphasis is on the model ⟷ symbols stage in the instructional sequence.



Children are asked to represent a concrete or pictorial model with symbols, and they are given a symbolic expression and asked to represent it with a drawing or a concrete model.

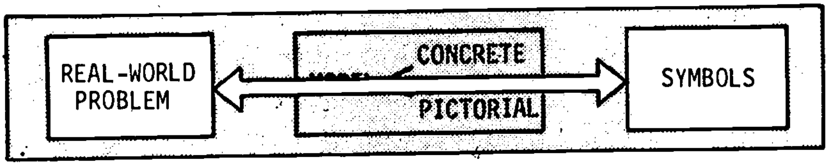




Teachers vary the activities, using games, pattern identification, drills, etc., to encourage children to learn and memorize the basic multiplication and division facts. Of course, the conceptual development of multiplication and division should continue throughout the second stage and into the third stage, which is algorithm development.

STAGE III: THE ALGORITHMS

The development of the algorithms for multiplication and division begins during grade 3 but receives most of its emphasis in grades 4 and 5. In this stage the real-world problem ⟷ symbols instructional point of view is emphasized.



The reason for the progression directly from the problem to the symbols is that it is often difficult to model the situation presented in the problem. For example, a third- or fourth-grade child might be given a problem such as

> Eight cases of coke were delivered to the school.
> Each case contains 24 bottles.
> How many bottles were delivered?

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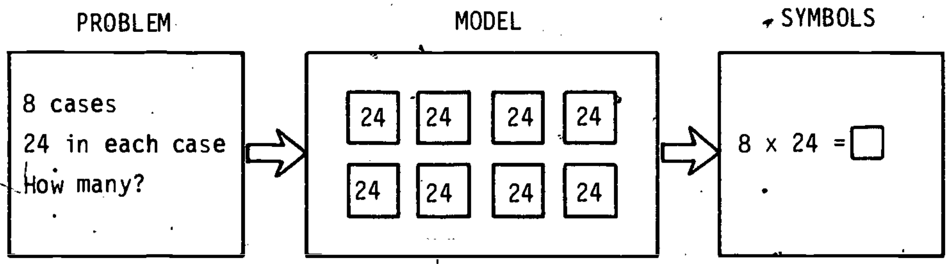
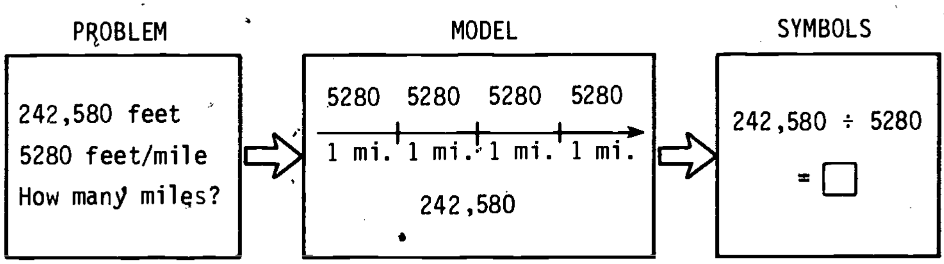
The numbers involved in this example are too large to be modeled with a symbol for each bottle. The model in this case, showing 24 x 8 bottles (and in other situations where the numbers are even larger), might actually be distracting. On the other hand, a more abstract model using some symbols can be very helpful. The nature of the models change as the numbers become larger.



This model relies less on objects and discrete drawings and more on symbolic drawings. Thus, in the grade-5 or -6 problem which follows, only a very general sketch is helpful.

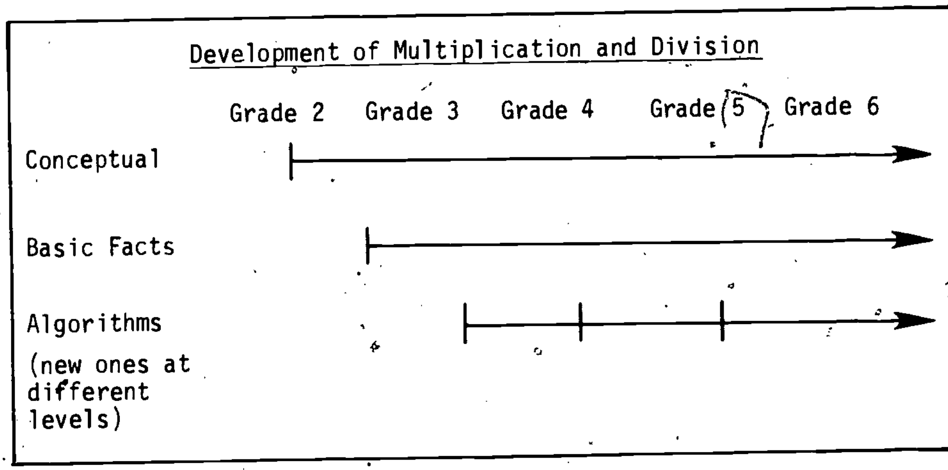
There are 5280 feet in a mile.
How many miles are there in 242,580 feet?



Notice that the model in this example is very sketchy and the emphasis remains on the translation from the problem to the symbols.

The development of the algorithms proceeds as one might expect--easiest to hardest. The easiest algorithms involve examples with few digits (24 x 8 or 125 ÷ 5), while the harder examples have many digits (435 x 628 or 14253 ÷ 684).

It is important to conclude with a review of two major ideas. The first is that in teaching and learning mathematics it is best to proceed from a real-world problem to a model to the symbols. The second is that there are three stages in the development of multiplication and division: the conceptual stage, the learning of basic facts, and the learning of algorithms. Each of the three stages starts at a different grade level but all three continue through the elementary school years.



ACTIVITY 2

INTRODUCING MULTIPLICATION AND DIVISION: USING PROBLEMS ⟶ MODELS ⟶ SYMBOLS

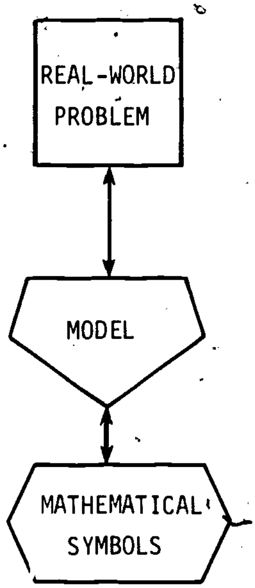
FOCUS:

This activity will present several examples which can serve to introduce multiplication and division to children. The examples are presented using the real-world problem ⟶ model ⟶ symbol strategy.

DISCUSSION:

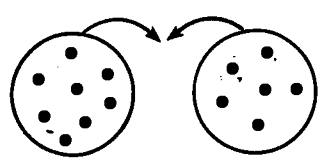
Throughout the Mathematics-Methods Program the relationship between the real world and mathematics is constantly emphasized. Specifically, in developing addition of whole numbers with children, this relationship can be illustrated as follows:

The statement of the problem.



A club has 8 members. 6 new members join the club. How many are now in the club?

The construction of a model, which may use either concrete materials or a picture.



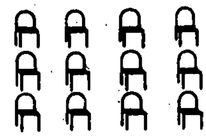
The translation of the problem into mathematical symbols.

$8 + 6 = \square$

This diagram illustrates the relationship that is built between the real-world situation and the mathematical symbols. The arrows in this example show the flow from the real-world situation to the mathematics and in the other direction as well. That is, if a child is given a mathematical sentence $3 \times 4 = \square$ for example, he or she should be able to construct a model and make up a problem for it.

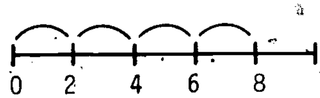
There are four different models used in the development of multiplication and division. They are arrays, sets, measure (number line), and cross products. The first two are closely related, their difference being more in the physical arrangement than in the concept. Each is briefly illustrated below.

a) Array — 3 rows of chairs, 4 in each row



b) Sets — 5 bags of marbles, 3 in each bag

c) Measure (number line) — 2 feet to a stride, 4 strides



d) Cross Product — 4 ice creams (chocolate, vanilla, strawberry, peppermint)
3 toppings (fudge, butterscotch, marshmallow)

How many sundaes with one ice cream and one topping?

TOPPINGS

ICE CREAM	F	B	M
C	C/F	C/B	C/M
V	V/F	V/B	V/M
S	S/F	S/B	S/M
P	P/F	P/B	P/M

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DIRECTIONS:

1. Ten examples follow: examples 1 - 4 suggest multiplication; examples 5 - 7 suggest division; and examples 8 - 10 suggest multiplication or division or both. Each example contains either a problem or a model or a mathematical sentence. You should supply the missing parts. Complete the examples on your own.

2. Compare and discuss your work with other members of your group.

   a) For each example characterize the model used as set, array, measure, or cross product.

   b) Which models seem to be more helpful?

   c) Could multiplication (or division) sentences be written in place of the division (multiplication) sentences you wrote? What implications might this have for you as you are teaching children?

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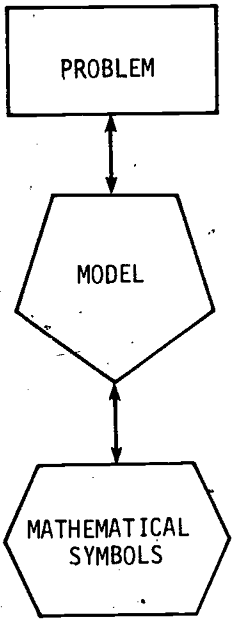
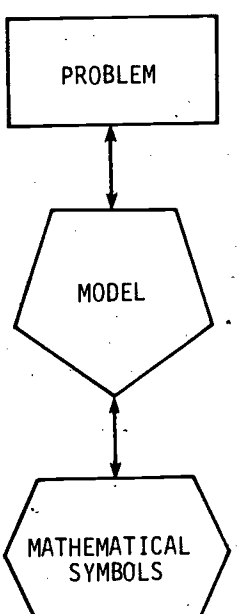
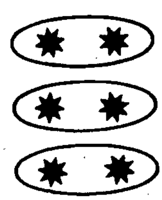
Example 1



There are three rows of mailboxes for an apartment house. Each row contains five mailboxes. How many mailboxes for the apartments?

Example 2





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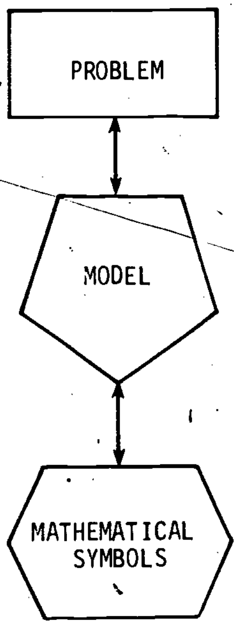
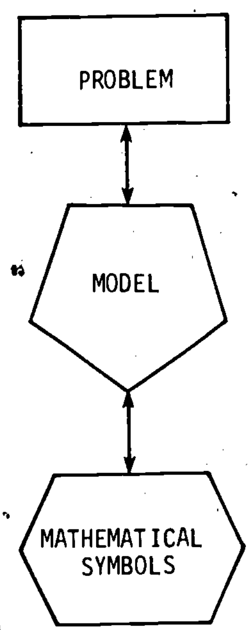
Example 3:



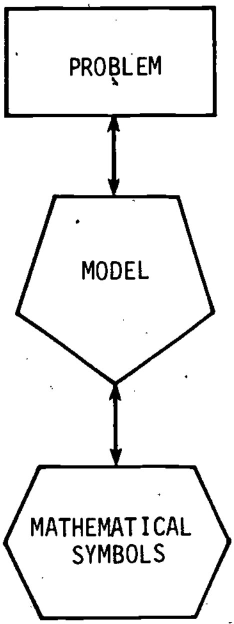
An ice cream shop has six flavors and two kinds of topping. How many different combinations for a sundae are available?

Example 4:



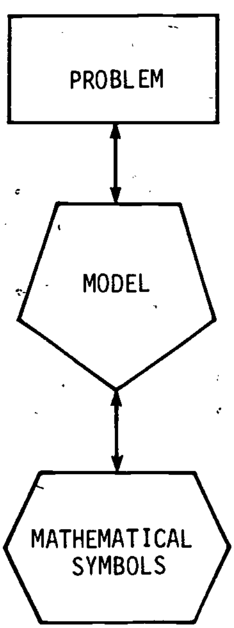
$3 \times 4 = \square$

Example 5:

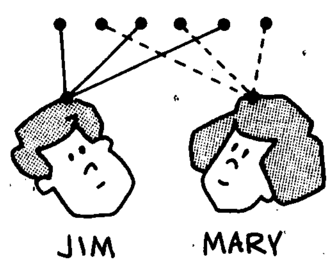


Eighteen people in a class are formed into teams of six people each. How many teams are there?

Example 6:



One for Jim,
One for Mary,
One for Jim,
One for Mary,
One for Jim...



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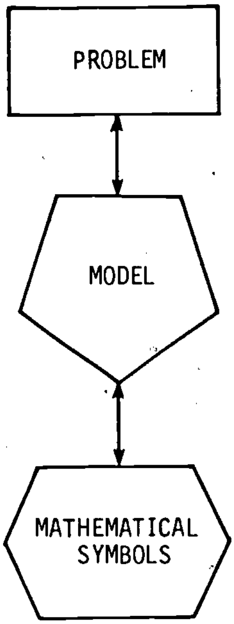
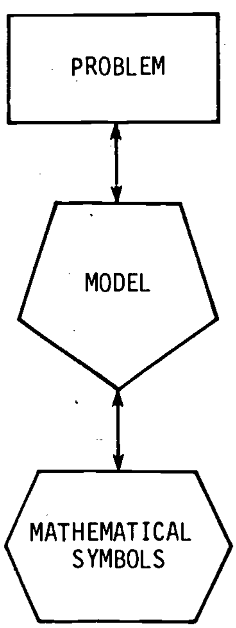
Example 7:



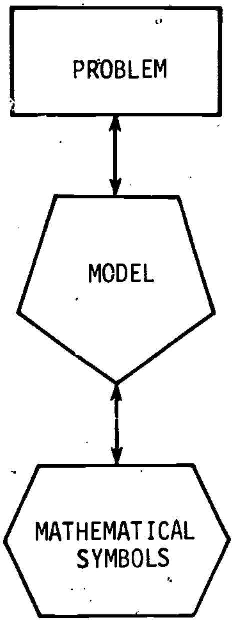
$2 \times \square = 14$

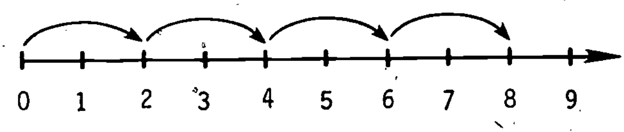
Example 8:



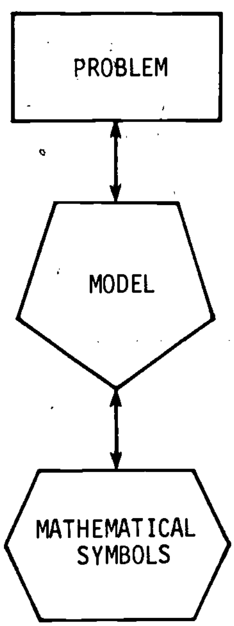
Mr. Brown has a log that is 18 feet long. How many three-foot pieces of wood can he obtain from this log?

Example 9:





Example 10:



PANTS	SHIRTS: Red	Blue	Yellow	White
Black	B/R	B/Bl	B/Y	B/W
Brown	Br/R	Br/Bl	Br/Y	Br/W
Gray	G/R	G/Bl	G/Y	G/W

ACTIVITY 3

SEQUENCING INITIAL CONCEPTUAL WORK IN MULTIPLICATION AND DIVISION

FOCUS:

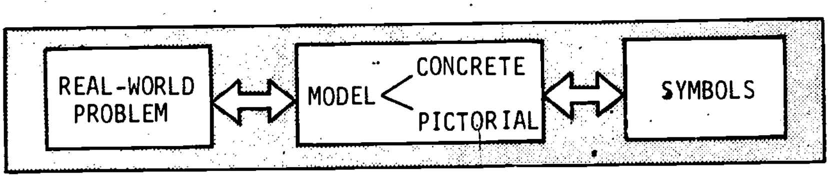
The previous activity focused on some examples of initial introductory work in multiplication and division for young children. In this activity you will be asked to sequence a broader set of experiences in the conceptual development of multiplication and division for children.

MATERIALS:

Elementary school mathematics textbooks, grades 3 and 4 (one set per group).

DISCUSSION:

The instructional point of view displayed below is one that should be maintained throughout all three stages of the development of multiplication and division, i.e., introduction, basic facts, and algorithms. As you proceed through this activity keep this point of view in mind and look for ways in which each exercise might be introduced using this point of view.



DIRECTIONS:

1. Remove the card set from the unit and separate the cards. Eleven illustrations of exercises related to the introduction of multiplication and division are depicted on the cards. <u>Without using the textbooks</u>, number these exercises according to the order in which you think they ought to be presented to children.

a) First do it individually.

b) Then discuss your sequence with a partner or in a small group. Record the sequence you agree on.

c) Study the grade 3 and 4 textbooks provided. Record the textbook sequence. Compare it with the sequence your group thought appropriate. In the seminar (Activity 6) you will have a chance to discuss the strengths and weaknesses of both orderings.

Your Sequence	Group Sequence	Textbook Sequence
______	______	______
______	______	______
______	______	______
______	______	______
______	______	______
______	______	______
______	______	______
______	______	______
______	______	______
______	______	______
______	______	______

2. Identify (if possible) introductory experiences that you feel are important (or that you observed in the textbook) and which are not included in the card set. Record these experiences for a future class discussion.

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ACTIVITY 4

THINKING ABOUT MULTIPLICATION AND DIVISION

FOCUS:

In this activity you will have an opportunity to consider ways in which children might think about solving multiplication and division examples. This activity also contains a discussion of two division situations, partitive and quotative.

DISCUSSION:

Once multiplication and division have been introduced through real-world examples, one has to proceed in the direction of formalization. At the same time, too early a formalization with meaningless symbols is unsound. How then should one help children solve examples such as $3 \times 4 = \square$ and $15 \div 3 = \square$? The processing of these number sentences should proceed directly from the problems and the models used to introduce the operation. This activity gives a series of illustrations of children solving some multiplication and division examples.

DIRECTIONS:

1. Many different strategies are used by children to solve multiplication and division examples when they have no formal procedures for solving them. They use whatever technique they have at hand. Some of these techniques are:

   - repeated addition
   - skip counting
   - counting
   - repeated subtraction
   - counting backwards
   - multiplication (for division)

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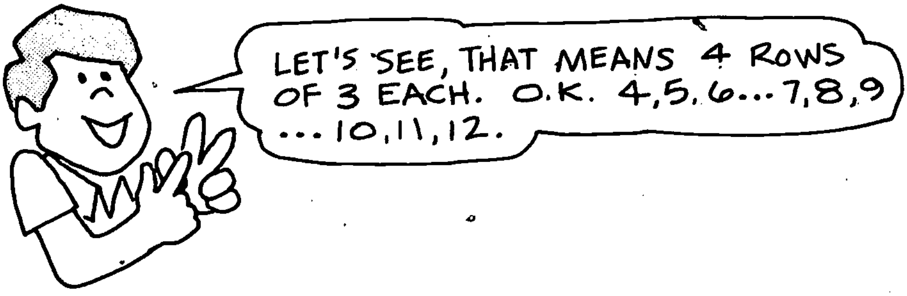
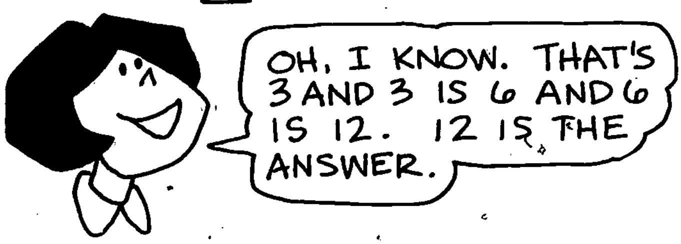
Study each of the illustrations below and identify which strategy the child might be using in solving the example.

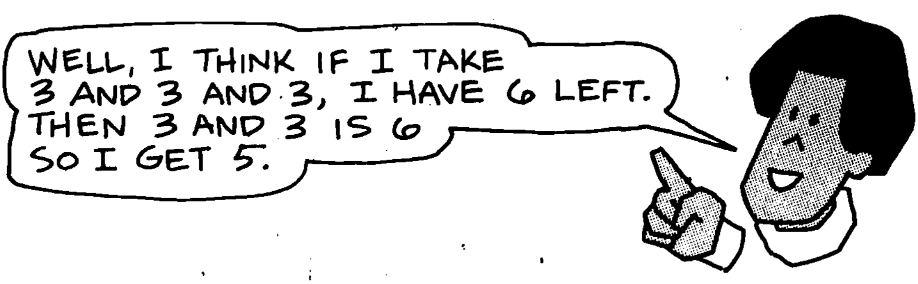
a) John: $4 \times 3 = \square$



b) Susan: $4 \times 3 = \square$



c) Phil: $15 \div 3 = \square$



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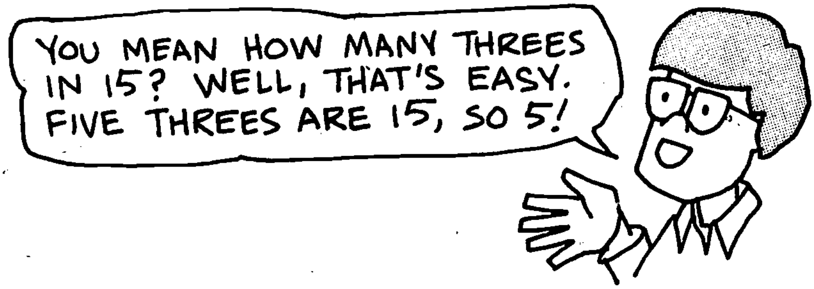
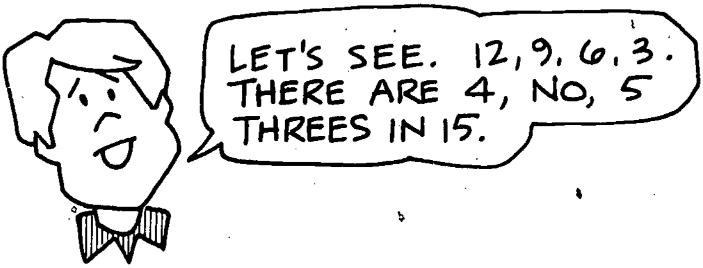
d) Lisa: 3 x 4 = $\square$

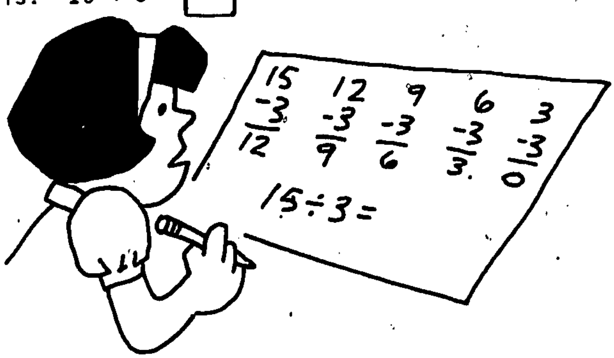
e) Mark: 15 ÷ 3 = $\square$



f) Jimmy: 15 ÷ 3 = $\square$



g) Phyllis: 15 ÷ 3 = $\square$



2. Why is it important to encourage children to solve early multiplication and division examples using their own strategies? How might you encourage such thinking?

3. How might you solve such examples if you had no training other than that of a second-grade child? Make a list of all the strategies you and your classmates can think of.

DISCUSSION:

There are two situations which a division sentence might represent. We can see the difference between these two situations by looking at two examples for the division sentence 12 ÷ 4 = 3. In the quotative situation we know the number of elements in each set and we ask how many sets there are. For example:

John has 12 pieces of candy.
He decides he will give three pieces to each of his friends.
To how many friends does he give his candy?

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In the partitive situation we know the number of sets and we ask how many elements there are in each set. For example:

John has 12 pieces of candy.
He decides he will give it to three friends.
How many pieces does each friend get?

Although it is not useful to use the terms "quotative" and "partitive" with children, they do need to learn that both kinds of situations are represented by division sentences. As a teacher you need to understand the difference between the two situations so that you can analyze children's work to be sure that they develop the ability to apply division sentences to both situations.

DIRECTIONS:

4. Tell which of the following examples represent partitive situations and which represent quotative situations.

   a) Phil has 18 Coke bottles. How many carriers of six bottles can he fill with these bottles?

   b) Phyllis sold 45 boxes of cookies. She sold the same number on each of five days. How many boxes did she sell each day?

c) Pete collects stamps. He has 24 to display on one page. If he puts four in each row, how many rows will he have on the page?

d) Phoebe wants to buy 30 candies for her classmates. Candies are sold six in a package. How many packages must she buy?

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ACTIVITY 5

DEVELOPING DIVISION WITH REMAINDERS

FOCUS:

While the early development of all topics in multiplication and division should proceed carefully, division with remainders presents special problems.

DISCUSSION:

The child first learns about division by thinking of the missing factor in multiplication. For example, suppose the child wants to solve the problem, "Each car holds 4 scouts. We need to take 12 scouts. How many cars are needed?" He can think of $12 \div 4 = \square$ as $\square \times 4 = 12$. The problem can be conveniently written as either a multiplication or division sentence.

Many division situations, however, do not "come out even." Consider the problem "Each car holds 4 scouts. We need to take 22 scouts. How many cars are needed?"

Division using whole numbers presents another special problem since the answer may not be a whole number. For example, an older child will give the answer to $27 \div 5$ as 5.4 or $5\frac{2}{5}$, or some other equivalent form. For young children, however, such answers are meaningless since they have not studied fractions or decimals. To avoid this problem an example for young children should always be cast in some real-world situation in which an answer does have a meaning. Thus, 27 cents changed into nickels gives us 5 nickels and 2 cents remaining. Or 27 books and 5 boxes results in 5 books per box with 2 books left over.

Since children have not studied fractions the question arises, "Why introduce division with remainders?" Two considerations which might help to answer this question follow.

1) Long before rational numbers are taught, real-world situations arise for children which demand division with remainders.

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2) Some real-world situations arise in which a fractional answer doesn't make sense. An example of this is the 27 books and 5 boxes. It would not make sense to place $5\frac{2}{5}$ books in each box!

An instructional issue also arises in that without fractions one cannot reasonably write a division sentence for 27 ÷ 5. Rather one writes $27 = 5 \times \boxed{5} + \triangle\!2$. Fortunately, this number sentence coincides with the thinking of young children. For example, in 32 ÷ 6 one thinks, "How many sixes are in 32 and how many are left over?" $32 = 6 \times \square + \triangle$.

DIRECTIONS:

1. After studying the example and symbols at the right, complete the problems below which explore the equation

$$p = qd + r$$

42 ÷ 8

42 = 5 x 8 + 2

dividend (p), quotient (q), divisor (d), remainder (r)

Find <u>all</u> values for q and r which make the sentence true. Use only whole numbers.

a) $20 = (q \times 6) + r$

q	0							
r								

b) $37 = (q \times 5) + r$

q	0							
r								

c) $27 = (q \times 9) + r$

q	0							
r								

43

2. Which of the many possible pairs of r's and q's would you choose for the solution of the division problem related to the equations? What is the relationship between r and d in the pairs that you choose?

3. In (1) above, you notice that, in general, several values are possible for q and r. There needs to be a rule which determines which of the pairs represents the solution of the problem $p \div d$. This rule is summarized as follows.

> If p and d are whole numbers with $d \neq 0$, then there exist unique whole numbers q and r such that $p = (q \times d) + r$ where $0 \leq r < d$. The unique q and r are called the quotient and remainder for the division problem $p \div d$.

Complete each of these number sentences, so that r satisfies the above condition, and indicate the division problem represented by the number sentence.

$20 = (q \times 6) + r$ q = ________ r = ________ ____ ÷ ____

$37 = (q \times 3) + r$ q = ________ r = ________ ____ ÷ ____

$27 = (q \times 9) + r$ q = ________ r = ________ ____ ÷ ____

4. In subtraction the set of whole numbers is not closed since, for some a and b, $a - b$ is not a whole number. For example, $3 - 7$ is not a whole number. Are the whole numbers closed under division? Give examples.

5. Discuss why the point of view of proceeding from real-world problems to a model to symbols is particularly important for division with remainders. Summarize your discussion in a few sentences.

6. Complete the missing portions of the examples on pages 35 - 36.

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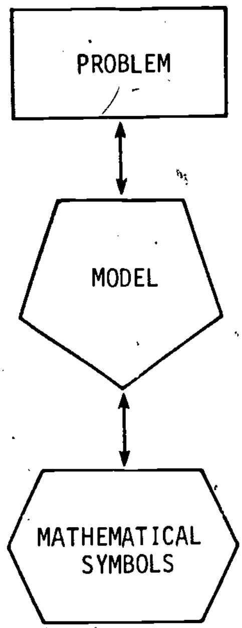
THE 17 CHIP GAME
(A Game with Remainders)

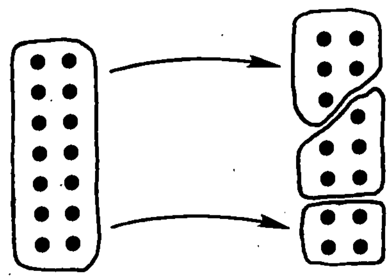
Here is a game. The winning strategy is given at the end of this activity but don't look yet! Play the game with a partner. See if you can discover a winning strategy. If not, one of you can read the winning strategy and see if you can help your opponent discover the strategy.

Rules:

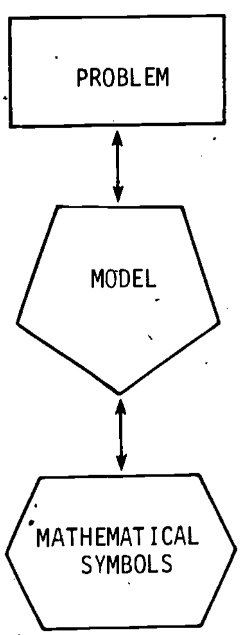
1. Spread out 17 chips (pennies, etc.)
2. Take turns in picking up chips. Each time you may pick up one, two, or three chips.
3. The person who picks up the last chip loses. (Remember you must pick up at least one and no more than three chips each time.)

Example 1:



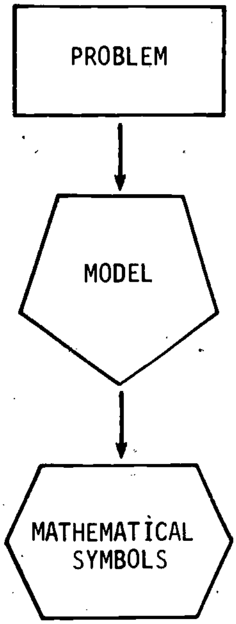


Example 2:



Mrs. Brown has baked 16 cookies. She wishes to divide them equally among Bill, Jane and Mary. How many will each child receive?

Example 3:



$12 = \boxed{2} \times 5 + 2$ or

$12 \div 5 = 2$ remainder 2

WINNING STRATEGY FOR THE 17 CHIP GAME

To Win: Let Your Partner Go First

If your partner picks up 1 you pick up 3.
If your partner picks up 2 you pick up 2.
If your partner picks up 3 you pick up 1.

Apply the above rules each time your partner picks up. Notice that the number picked up by your partner + the number picked up by you = 4 for each pair of turns.

Why Does It Work?

Consider the division of 17 by 4.

$$17 = (4\cdot 4) + 1$$

The game is arranged so that each time a pair of drawings has been made 4 chips have been removed from the pile. Since you pick up after your partner you are assured that:

$4\cdot 1 = 4$ have been removed after the first pair of drawings.
$4\cdot 2 = 8$ have been removed after the second pair of drawings.
$4\cdot 3 = 12$ have been removed after the third pair of drawings.
$4\cdot 4 = 16$ have been removed after the fourth pair of drawings.

This means that in your partner's next draw the final chip must be picked up.

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ACTIVITY 6

SEMINAR

FOCUS:

The purpose of the seminar is to clarify, synthesize, and summarize the material in Section I concerning the introduction of multiplication and division concepts to children.

DIRECTIONS:

The conceptual development of multiplication and division is basic to the future learnings of basic facts and algorithms (discussed in Sections II and III). The purpose of this seminar is to focus on issues and problems related to this conceptual introduction. Some questions follow which may help to start discussion.

1. What is meant by the "real-world problem → model → symbol" point of view? Give several examples.

2. Give several examples of instances with which children might be familiar in their lives which reflect multiplication and division.

3. Experience and research* suggest that the use of the word "share" helps young children develop the concept of division. Why would you suspect that this might be true? Can you suggest other words that might have a similar (if not such a strong) association with the division situation?

---

*Calvin J. Irons, "An Investigation into Second Grade Children's Ability to Solve Six Types of Division Problems Involving Sharing, Sharing-implied, and Non-sharing Situations" (Ph.D. dissertation, Indiana University, 1975).

4. Research* also indicates that children seem to be able to solve problems that involve action more readily than problems which do not imply action.

EXAMPLE OF "ACTION" ADDITION

Action: John has 4 pencils. He received 3 more for his birthday. How many does he now have?

Non-Action: John has 4 yellow and 3 green pencils. How many pencils does he have in all?

What implication (if any) does this have for multiplication and division?

*John F. LeBlanc, "The Performances of First Grade Children in Four Levels of Conservation of Numerousness and Three I.Q. Groups when Solving Arithmetic Subtraction Problems" (Ph.D. dissertation, University of Wisconsin, 1968).

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Section II

DEVELOPING THE BASIC MULTIPLICATION AND DIVISION FACTS

Section I focused on developing the concepts of multiplication and division, with emphasis on the relationship between the real world and mathematics. After being introduced to these operations, children are expected to learn certain skills related to these operations. Knowledge of basic number facts is one of the main goals. While memorization of these facts is desirable, teachers must take care that this is done only after appropriate experiences designed to make memorization meaningful. Further, as children begin to memorize these number facts, teachers should help them develop processes of mathematical thinking by having them use thinking strategies that rely on certain number properties and relationships among the facts. There is little question that experiences with thinking strategies such as pattern finding and reasoning with number properties are as important as the memorization of the facts themselves.

The diagram on the following page shows the thinking pattern development that a teacher should use to bridge the gap between the introduction of basic number facts and the memorization of those facts.

Conceptual Development	Getting Ready to Memorize	Memorization
1. The real-world problem 2. Using concrete or pictorial models 3. Writing mathematical sentences 4. Solving the mathematical sentences using the appropriate model	1. Using thinking strategies 2. Looking for and using number patterns 3. Using number properties and relationships	1. Drill activities 2. Games

This section focuses mainly on the "getting ready to memorize" step in bridging the gap between the conceptual development and memorization. Activities 7 and 9 focus directly on thinking strategies children use in organizing their memorization of basic facts. Activity 8 reviews the number properties and the relationship between multiplication and division from the perspective of their implications for helping children memorize the facts. Activities 10 and 11 focus on techniques and games appropriate for drill to help children memorize the facts.

MAJOR QUESTIONS

1. Discuss the role of using thinking patterns in helping children learn basic number facts. Cite two or three ways that you would encourage this in your classroom.

2. State four or five mathematical number properties that relate to the learning of the basic facts and give an example showing the utility of each.

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ACTIVITY 7

GETTING READY TO MEMORIZE BASIC FACTS

FOCUS:

In this activity some children's patterns of thinking related to basic number facts will be studied.

DISCUSSION:

The introduction to this section discussed the important step between the introductory concepts of multiplication and division and the memorization of the basic facts. This step may be called "getting ready to memorize" the basic facts. There are 100 basic facts for each of multiplication and division. Clearly, it is a burden for children to memorize these facts if they are viewed as independent facts, unrelated to one another. Some system, some organization has to be effected to help children memorize these facts. More important than the immediate goal of memorizing these facts, however, is the value of having children learn to think about number relationships and gain confidence in looking for number patterns.

The teacher plays a major role in this stage of the child's development through encouraging children to think about and reason out number combinations before memorization. At the same time, children seem to learn a great deal from other children as they hear them verbalize their thinking strategies.

DIRECTIONS:

1. If possible interview three or four children (ages 7 - 8), asking them some basic facts such as 4 x 3 or 7 x 6 or 56 : 8. As they respond, ask them how they got the answer or, if they don't know, how they might get the answer. Record the strategies used by the children.

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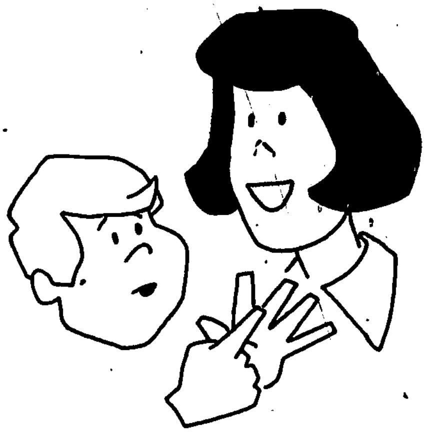
2. In a small group study each of the child responses given below and discuss the comments and questions with your classmates.

a) Teacher: How much is 3 x 4?
Mark: 12 (automatically)
Teacher: How much is 12 ÷ 4?
Mark: (Pause) Let's see... (using fingers) 8, 4, oh yes, 2? No, I mean 3.
Teacher: Are you sure?
Mark: (hesitatingly) I think so.



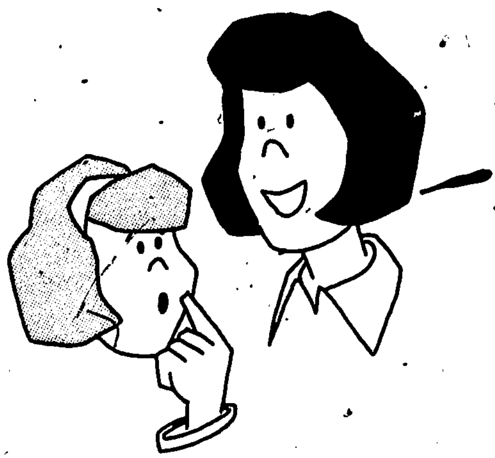
COMMENT

It seems that Mark does not have the relationship between multiplication and division well established in his mind. The inverse relationship is not only conceptually important but in this case quite useful.

QUESTION

How might you develop this inverse relationship for children like Mark?

b) Teacher: Bonnie, how much is 3 x 5?
Bonnie: 15
Teacher: How much is 7 x 5?
Bonnie: Let's see. (pause) 35?
Teacher: That's right. How much is 5 x 7?
Bonnie: Let's see. (pause) 42?
Teacher How much is 7 x 5 again?
Bonnie: 35



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Teacher: How much is 5 x 7?

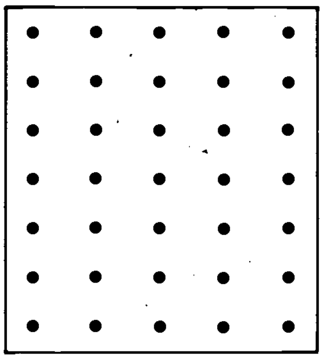
Bonnie: (squirming)...

COMMENT

Bonnie does not know the commutative property of multiplication (a x b = b x a) operationally. Bonnie probably knows her "5's" because children often learn to count by 5's (nickels, etc.).

QUESTION

How might you use an array card like the one shown here to help children like Bonnie gain an operational understanding of the commutative property?



c) Teacher: Betty, how much is 2 x 6?

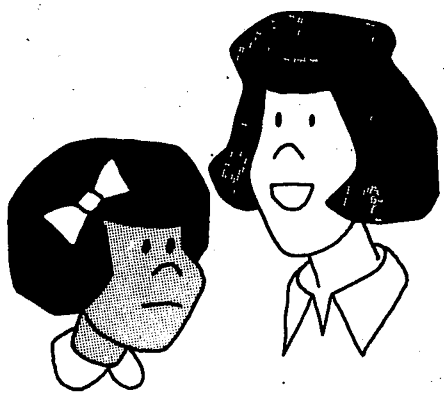
Betty: 12.

Teacher: How much is 4 x 6?

Betty: Hmmm...double 2 x 6, ...24.

Teacher: That's right. How about 8 x 6?

Betty: (confidently) 48!



COMMENT

Betty doesn't know it perhaps but she is using the associative property of multiplication (a x b) x c = a x (b x c). That is, she is thinking 4 x 6 = (2 x 2) x 6 = 2 x (2 x 6).

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QUESTION

How might you help another child see what Betty intuitively knows--at least for the "doubles"? Do you think Betty, knowing 2 x 6, might think of 6 x 6 as 3 x (2 x 6)? How might you help Betty understand that she can use "triples" as well as "doubles"?

d) Teacher: Kent, how much is 3 x 4?

Kent: 12.

Teacher: O.K. How much is 4 x 4?

Kent: 16

Teacher: Right. How much is 7 x 4?

Kent: Let's see. (mumbling) 12 and 16. Oh yes, 28.

COMMENT

Kent is intuitively using the distributive property of multiplication over addition: (a + b) x c = (a x c) + (b x c). In this example he reasons that 7 = 3 + 4 so

$$7 \times 4 = (3 + 4) \times 4 = 3 \times 4 + 4 \times 4 = 28.$$

QUESTION

How can this strategy be useful to children who know some simpler facts but do not know the greater number facts such as 7 x 8?

e) Teacher: David, how much is 3 x 6?

David: 18.

Teacher: Okay! How much is 4 x 5?

David: 20.

Teacher: Good. Now how much is 5 x 8?

David: (silence)... I don't remember that one. Is it 13?

COMMENT

David is typical of many students who have simply tried to memorize every fact. His answer for 5 x 8 shows no insight or knowledge of the number facts. He is in obvious need of help to develop some thinking strategies. As a starting point, he does not even seem to have any concept of order, since he answers 20 for 4 x 5 and 13 for 8 x 5.

QUESTION

What questions might you ask David which would help him both to develop thinking strategies and to get the answer to 5 x 8? What might happen to David in his mathematical development if an emphasis on thinking strategies is not fostered?

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ACTIVITY 8

PROPERTIES OF NUMBERS IN MULTIPLICATION AND DIVISION

FOCUS:

In this activity you will have a chance to review some properties and terms related to multiplication and division.

MATERIALS:

Elementary school mathematics textbooks.

DISCUSSION:

In Activity 7 you read and discussed anecdotes in which knowledge of certain mathematical relationships and properties was useful in thinking about basic number facts. This activity is divided into two parts: Part 1 provides an opportunity to review some basic number properties and relationships; and Part 2 provides an opportunity for you to identify some practical applications of these properties and relationships to classroom learning, specifically to the learning of basic number facts.

DIRECTIONS:

PART 1

1. For each term write a statement or a definition and provide an example. You might find the statement or definition in a high school or college mathematics textbook, but you should use an elementary textbook for the example.

Zero Property of Multiplication	Example	$3 \times 0 = 0$
	Statement	The product of zero and any whole number is zero. That is, $a \times 0 = 0$ where a is a whole number.
Factor	Example	
	Definition	
Commutative Property of Multiplication	Example	
	Statement	
Closure Property of Multiplication	Example	
	Statement	
Multiple	Example	
	Definition	

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Distributive Property of Multiplication over Addition	Example	
	Statement	
Multiplicative Identity	Example	
	Statement	
Product	Example	
	Definition	
Associative Property of Multiplication	Example	
	Statement	
Divisor	Example	
	Definition	

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Dividend	Example
	Definition
Quotient	Example
	Definition
Remainder	Example
	Definition
Multiplicative Inverse	Example
	Statement
Division Inverse	Example
	Statement

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2. For each of the properties listed write T for true or F for false for the operations of multiplication and division, considering the set of whole numbers only.

	Multiplication	Division
Commutative		
Associative		
Distributive		
x over +		
÷ over +		
Identity Element		
Inverse		
Closure		

Discuss your answers with fellow classmates. Record questions for your instructor for the seminar which concludes this section.

PART 2

Complete the multiplication table which appears on page 53 and use it to answer the division questions which follow it.

x	0	1	2	3	4	5	6	7	8	9
0										
1			2							
2						10				
3										
4										
5										
6								42		
7										
8										
9										

1. a) Write 7 x 8 = 56 as a division sentence (two ways).

   b) Division is said to be the inverse of multiplication. Tell how this relationship could be used as a strategy to help children reduce the number of basic facts they must memorize.

2. How many of the 100 multiplication facts could be eliminated from memorization by using the commutative property? Give examples.

3. How can the associative property be useful in reducing the number of multiplication facts?

4. How is the identity element for multiplication useful in the memorization of multiplication and division facts?

5. How many of the 100 multiplication facts could be learned through knowing the zero property of multiplication?

6. How is the distributive property useful in learning new basic facts from previously learned facts?

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ACTIVITY 9

SEARCHING FOR STRATEGIES

FOCUS:

In this activity you will have an opportunity to summarize some patterns and thinking strategies related to the memorization of basic facts.

DIRECTIONS:

1. The objective of this activity is for you to fill out the chart on page 56 with as many helpful patterns and strategies as possible. This chart should serve as a record for your future teaching experience. To fill out this chart, do several or all of the following:

   a) interview children

   b) interview teachers

   c) interview your classmates

   d) interview your spouse or date

   e) interview yourself.

   Jot down all interesting ideas. For example, in discussing multiples of 5 some observations might be "always ends in 0 or 5," "I think about nickels," "I count by 5's," etc.

2. Decide whether the following multiplication and division statements are true (always) or false. If the statement is true, state the property illustrated. If the statement is false, give a numerical example which shows that it is false. The letters <u>a, b, c stand for any whole number</u>.

   T F 1. $a \times a = a$

   T F 2. $a \times (b \times c) = (a \times b) \times (a \times c)$

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T F 3. $0 \div a = 0 \quad (a \neq 0)$

T F 4. $a \div b = b \div a$

T F 5. $(a \times b) \times c = a \times (b \times c)$

T F 6. $a \times (b - c) = (a \times b) - (a \times c) \quad (b \geq c)$

T F 7. $(a + b) \div c = (a \div c) + (b \div c)$

T F 8. $1 \times a = a$

T F 9. $a \div 0 = 0$

T F 10. $a \div (b + c) = (a \div b) + (a \div c)$

TABLE

Multiples of:	List of patterns or strategies (two or three each) that might be suggested to help pupils find or recall the number facts associated with each number
0	
1	
2	
3	
4	
5	
6	
7	
8	
9	
10	

ACTIVITY 10

USING THE HUNDRED'S BOARD TO DEVELOP THINKING STRATEGIES

FOCUS:

You know that thinking strategies can be developed through familiarity with number properties and patterns. These strategies can also be aided by various geometric patterns on the hundred's board. The uses for the hundred's board seem endless, but in this activity your interest will be narrowed to concentrate on the number patterns, viewed by considering multiples, which are useful for the development of thinking strategies.

DISCUSSION:

The hundred's board is a board (or chart) on which the numbers 1 to 100 are arranged in 10 rows of 10 numbers. This board has many uses for work in numeration, addition and subtraction, and multiplication and division.

Using this board a child may be helped to see the order of numbers, as well as the geometric patterns formed by multiples of numbers. The example shown below illustrates the diagonal pattern that results from circling the multiples of 3.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

DIRECTIONS:

1. Use the four hundred's charts that follow to explore patterns formed by circling the multiples of 2, 4, 5 and 9.

Multiples of 2

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Multiples of 5

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Multiples of 4

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	94	96	97	98	99	100

Multiples of 9

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	[illegible]	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

2. Study the hundred's charts in (1) above. List four discoveries you might help children make in their study of multiples displayed on a hundred's board.

3. On the chart marked with the multiples of 9 mark the multiples of of 3 and 6 with a different colored pen or with △ and □ rather

than ◯. Which multiples of 3 are not multiples of 6 or 9? Which numbers are multiples of 3 and 6 and 9?

4. Write two activities using the hundred's chart which could help children see patterns useful in memorizing the basic multiplication and division facts.

TEACHER TEASER

Eight pirates decide to divide their gold. Each receives an equal amount but two pieces remain. How many pieces of gold were there?

Being greedy, two of the pirates fight over the two extra pieces. One is killed and the remaining seven divide his gold so that there are now five pieces of gold left over. Can you now tell how many pieces of gold there were totally? Is this answer unique?

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ACTIVITY 11

BUILDING SKILL USING GAMES

FOCUS:

Games are often useful in the memorization of basic facts. This activity will provide a sample of a few games and provide an opportunity for you to construct your own.

MATERIALS:

Listed separately for each game.

DISCUSSION:

Games in mathematics are often praised and often damned! Memorization of basic facts is a dull assignment, and games often provide a needed incentive for children to memorize these facts. Thus, games are promoted. The games prove to be so much fun that soon, unless (s)he is careful, the teacher is pressed to "entertain" the children with more games. Some teachers yield to this very real pressure and provide games, even games which have no specific educational or instructional purpose. Games, like any other instructional activity, should have specific objectives.

DIRECTIONS:

1. In your group, play the three games which are described on the following pages. After playing each game, decide how you would use it in the classroom. What is your view of the role of games in general for building skills?

2. Individually, design a game which could be used to promote skill development. State the rules and the objective for this game.

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GAME 1

"Watch Your P's and Q's!

MATERIALS:

Heavy paper, markers, scissors.

DIRECTIONS:

1. Make a set of 20 cards. Each card should have one numeral written on it. For early work with division the twenty cards should consist of two cards with each of the numerals 0 through 9 (i.e., 0, 0, 1, 1, 2, 2, ..., 8, 8, 9, 9).

2. Turn the cards face down.

   Each person selects three cards and forms all possible combinations of the cards with one card being the divisor, another the quotient, and the third the remainder.

   For example, if the cards 5, 2, and 7 were drawn, the possible combinations are represented in the table below.

	Divisor (D)	Quotient (Q)	Remainder (R)
a)	5	2	7
b)	5	7	2
c)	2	5	7
d)	2	7	5
e)	7	2	5
f)	7	5	2

3. Each player must then decide if each situation is legal and, if it is, what the dividend is. In the example above, (a), (c), and (d) are not legal (because the remainders are larger than the divisors), and the dividends for (b), (e), and (f) respectively are 37, 19, and 37.

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4. The dividends are added up. (37 + 37 + 19 = 93). The player with the highest score wins.

GAME 2
"Neighboring Squares"

DIRECTIONS:

On the array of whole numbers below, draw either one or two line segments through any two or three adjacent squares so that the product of the numbers in those squares is 120.

Here, we use the term "adjacent" to mean "any pair of squares which touch in any manner--either a common side or vertex." For example, the squares 1, 2 and 60 are adjacent and the product is 120. The game consists of locating all such sets of adjacent squares.

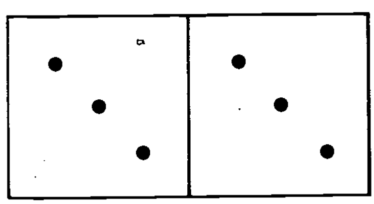
24	20	1	3	40	60
3	5	3	30	1	5
40	1	24	2	4	6
60	2	3	1	5	12
30	1	20	3	6	10
2	6	10	4	2	1
60	40	3	1	20	24

Variations of this game can be made by using different arrays, different products, or even different operations.

There are 18 cookies to be shared among 6 children.

How many cookies should each child receive?

A



How many sets of three? ______

2 threes = ______

2 X 3 = ______

B

MULTIPLICATION AND DIVISION
CARD SET

MULTIPLICATION AND DIVISION
CARD SET

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Find the greatest number for □.

Then find △.

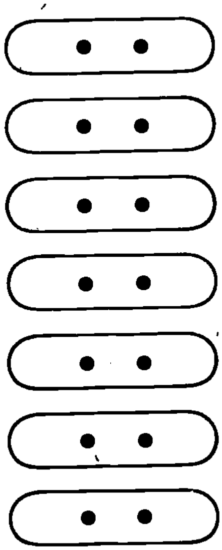
A. 8 = (1 X 5) + △.

B. 26 = (□ X 5) + 1.

C. 27 = (□ X 5) + △.

D. 35 = (□ X 5) + △.

C



Use this picture to complete the following sentences.

□ X 2 = 14

14 ÷ 2 = □

D

MULTIPLICATION AND DIVISION
CARD SET

MULTIPLICATION AND DIVISION
CARD SET

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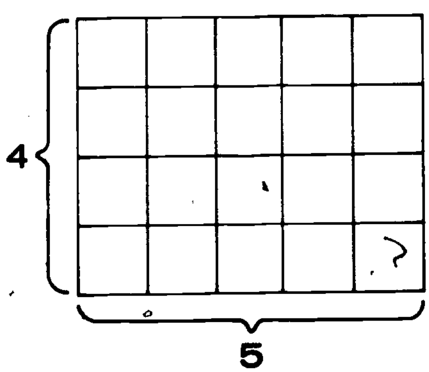
Candy bars are on sale for 8 cents each.

John bought 6 candy bars.

How much money did he spend?

E

Write two multiplication and two division sentences suggested by this array.



F

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MULTIPLICATION AND DIVISION
CARD SET

MULTIPLICATION AND DIVISION
CARD SET

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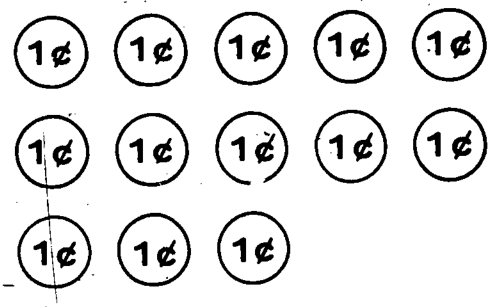
Cameron can get 1 nickel for five pennies.

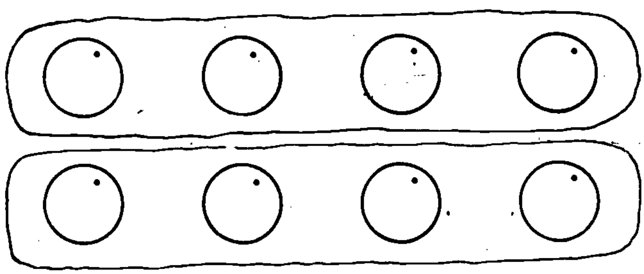


13 is two fives and 3 ones left over.

$13 = \square \times 5 + \triangle$

How many nickels will Cameron have?

G



A set of 8 oranges

____ sets of 4 are 8

____ X 4 = 8

H.

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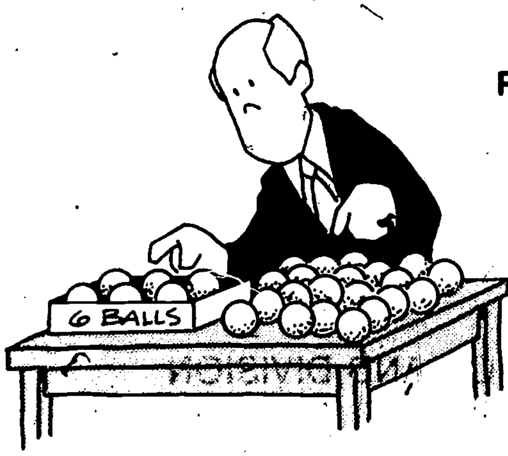
MULTIPLICATION AND DIVISION
CARD SET

MULTIPLICATION AND DIVISION
CARD SET



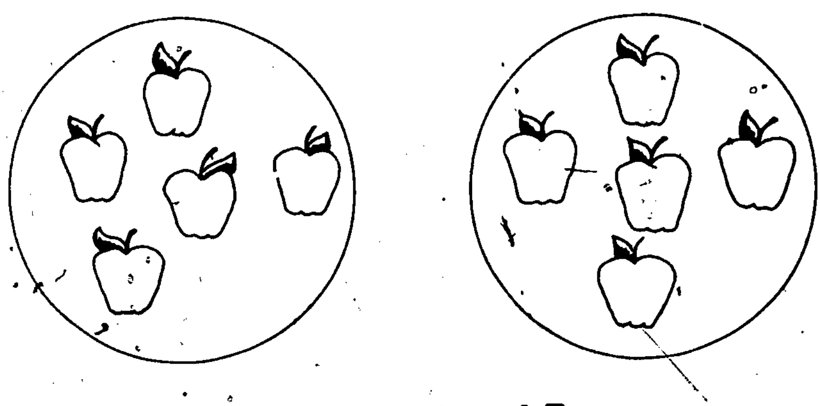
Ping-pong balls are packed 6 to a box.

Mr. Jones has 27 ping-pong balls.

How many full boxes of ping-pong balls will Mr. Jones have?

How many balls will be left over?

I.



____ sets of 5

Join the sets $5 + 5 =$ ____

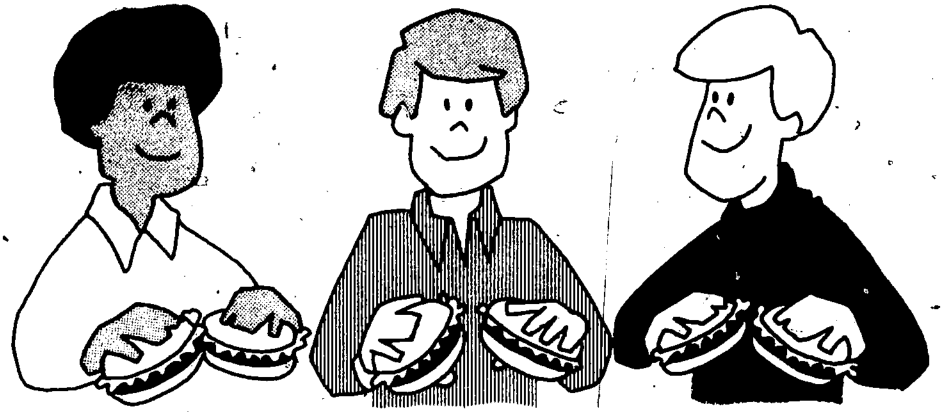
2 fives = ____

J

MULTIPLICATION AND DIVISION
CARD SET

MULTIPLICATION AND DIVISION
CARD SET

ERIC



Three boys

Two hamburgers each

How many hamburgers ?___ X ___ = _____

K

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MULTIPLICATION AND DIVISION CARD SET

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GAME 3

"The Cover-Up"

MATERIALS:

Construction paper or 1" graph paper, three dice, and a supply of playing pieces for each player.

DIRECTIONS:

1. Prepare a game board like the one shown on page 64. You will also need three dice and a supply of different markers for each player to cover the numbered squares.
2. To begin play, each player in turn rolls all three dice and the player with the smallest sum begins play. Play progresses around the table.
3. The first player rolls the three dice. He must use one or two operations on the three numbers shown on the dice. For example, suppose he rolls a 3, 4, and 5. He could think (3 x 4) + 5 or (3 + 4) x 5 or (3 x 4) - 5 or 3 x 4 x 5, etc. He is then allowed to cover one resulting number on the board with a marker (e.g., 60 = 3 x 4 x 5). The next player takes his turn. He may <u>not</u> cover a number already covered.
4. Several variations for scoring can occur. The simplest is to count the markers each person has at the end of the game. Another variation on the game is to allow only markers adjacent to markers already played.

$$X \left(\begin{smallmatrix}+\\-\\\times\\\div\end{smallmatrix}\right) Y \left(\begin{smallmatrix}+\\-\\\times\\\div\end{smallmatrix}\right) Z$$

1	2	3	4	5	6	7	8
9	10	11	12	13	14	15	16
17	18	19	20	21	22	23	24
25	26	27	28	29	30	31	32
33	34	35	36	37	38	39	40
41	42	44	45	48	50	54	55
60	64	66	72	75	80	86	90
100	108	120	125	144	150	180	210

ACTIVITY 12

SEMINAR

FOCUS:

This seminar provides an opportunity not only to review the activities of this section but also to place the activities of this section in perspective with Section I and to prepare the way for Section III.

DISCUSSION:

The goal of this section has been to make you aware of the importance of helping children develop thinking strategies to aid in the memorization of basic number facts, and to provide some techniques and insights to aid children in developing such strategies. In addition, some activities and games designed to promote interest in developing skills in the basic facts were provided.

DIRECTIONS:

The following questions may help stimulate class discussion.

1. How do the mathematical number properties serve as useful techniques for helping children develop thinking strategies?

2. What are several strategies that children might employ to make the memorization of basic number facts easier and more efficient?

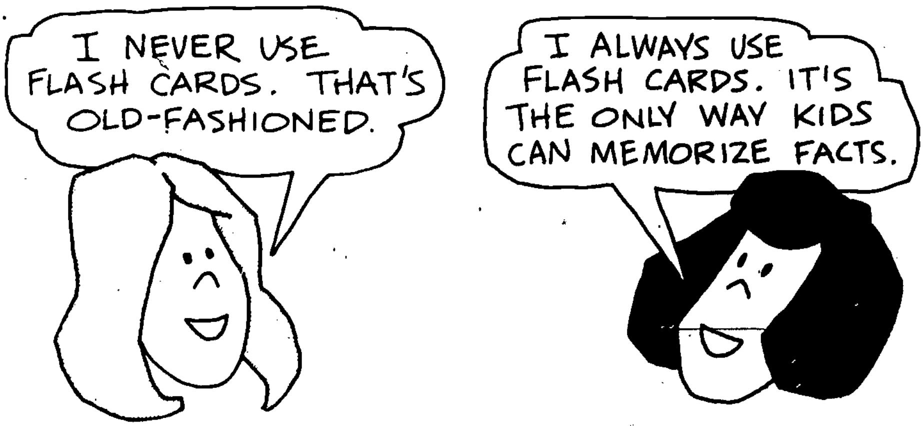
3. Assuming that all the properties are used, what might be the minimum number of basic multiplication facts to be memorized?

4. What are some activities that must precede the teaching of the basic multiplication and division facts?

5. Discuss the appropriateness of learning the multiplication facts in order; i.e., multiples of 0 first, multiples of 1 second, multiples of 2 third, and so on. What order might one use?

6. What is the appropriate role of drill and games in helping children remember basic number facts?

7.



Discuss the positions of these two teachers. What is your opinion on the flash card issue?

8. During the time of memorization of the basic facts, is there any need for conceptual development of multiplication and division?

9. Does the real-world problem ⟶ model ⟶ symbols point of view have any place in the teaching of basic facts?

10. Is it unusual for children in grade 4 not to know their basic facts? Grade 6? Grade 8? Grade 10? Sophomores in college?! What might be the causes for the lack of this skill in these individuals?

Section III

COMPUTATIONAL ALGORITHMS FOR MULTIPLICATION AND DIVISION

The previous two sections of this unit have focused on introductory work in multiplication and division and on the development of thinking strategies to help children understand the operations and acquire essential skills. An understanding of basic concepts (as well as some skill in using these concepts) is necessary for work with multiplication and division algorithms. You may recall that algorithms are efficient procedures for calculating (e.g., the long division algorithm).

Initially children develop an understanding of these algorithms by working with smaller numbers. In this way concrete aids or pictures can be used which model the real-world situation and the mathematics associated with it. As the numbers become larger, it is more difficult to use objects or pictures. They often would actually serve to confuse rather than help. (Imagine trying to use objects for 13 x 439.) So work with symbols alone begins. This symbolic work has intrinsic merit for children since one of the beauties of mathematics lies in the facility and power of its symbolism.

This section builds upon the concepts and skills developed in previous sections to develop algorithms or computational procedures for processing larger numbers. Just as in the teaching of the concepts, the development of the algorithms should proceed by relating real-world situations to mathematics through models. In the development of algorithms, as in the development of basic number facts, the

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properties of mathematics play an important role and should be utilized.

The development of the multiplication and division algorithms involves "fitting together" concepts and principles on which the algorithms are based. Thus, careful, sensitive teaching is required to build from basic conceptual learning to the procedures used in the algorithms. Teachers must also carefully monitor children's work in order to identify individual difficulties that may arise. As is often the case in this area, pupil performance at more difficult levels of the algorithms (e.g., 24 x 37 or 3749 ÷ 24) rests upon pupil insight and performance at lower stages of the algorithms. Because approaches to these algorithms vary somewhat among textbooks, teachers must understand the underlying concepts, be able to identify how an instructional sequence is developed, and be aware of issues related to different approaches.

This section will be divided into two clusters of activities; one dealing with the multiplication algorithm, the other with the division algorithm. Following these sets of activities will be an activity designed to help you identify pupil errors in multiplication and division and prescribe remedial experiences. It should be noted that children do not learn multiplication in isolation from division. Rather, the algorithmic development for each proceeds simultaneously in the school.

MAJOR QUESTIONS

1. What skills are prerequisite for a multiplication problem such as 78 x 243?

2. Outline the sequence of division examples you would present to a child, starting from basic facts and working toward an example like $52\overline{)4823}$.

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ACTIVITY 13

USING MODELS TO INTRODUCE THE MULTIPLICATION ALGORITHM

FOCUS:

This activity will challenge you to perform some tasks, both to acquaint you with various techniques and to suggest appropriate procedures for use with children. The child should already have had considerable experience with physical objects and pictures used in the development of the introductory concepts and the basic facts. Initially these same aids could be used in the development of the algorithms.

MATERIALS:

Dienes blocks, bundling sticks grouped in tens and ones; graph paper ($\frac{1}{4}$" grid is appropriate); Cuisenaire rods (optional).

DISCUSSION:

The use of pictures and physical aids is an essential ingredient in teaching elementary school mathematics. In Section I the instructional point of view of proceeding from a real-world problem to a model to symbols was stressed. In the Overview (Activity 1) it was pointed out that in the algorithmic development stage, emphasis would be placed on the direct progression from real-world problems to symbols since it would be difficult to model situations involving large numbers. Thus, any utilization of models must occur at the beginning of the algorithmic development when introducing the algorithms with smaller numbers. The purpose in using a model in introducing algorithms is both to help children "see" the reasonableness of the algorithmic processes and to help the children become confident of the symbolic representation. The children must be gradually "weaned" from pictures or aids which represent the actual number of objects and accustomed to using more symbolic models.

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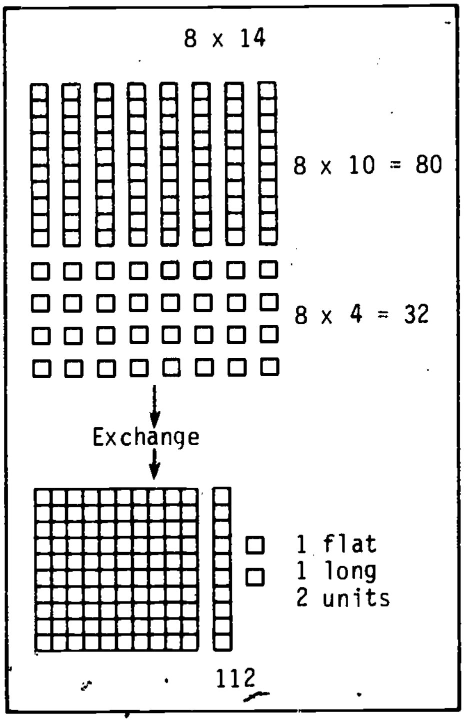
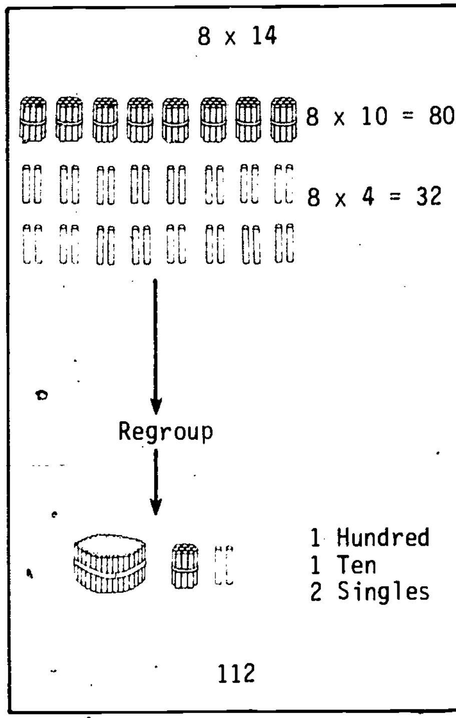
Example: $8 \times 14 = \square$

Initially aids could be used.

Dienes Blocks



Bundling Sticks



Children can be shown the relationship between the aids and the symbols $8 \times 14 = 8 \times (10 + 4) = 8 \times 10 + 8 \times 4$.

a) Where do we have 80?

b) Where do we have 32?

c) Where do we have 112?

As the numbers become larger, more abstract representations can be substituted for the use of pictures or aids which represent the actual objects.

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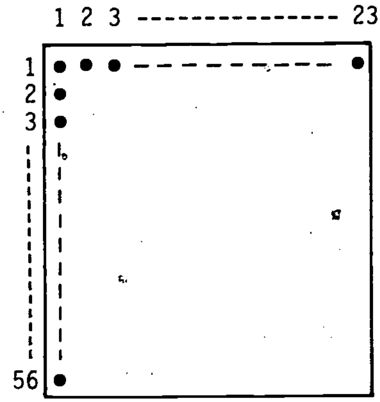
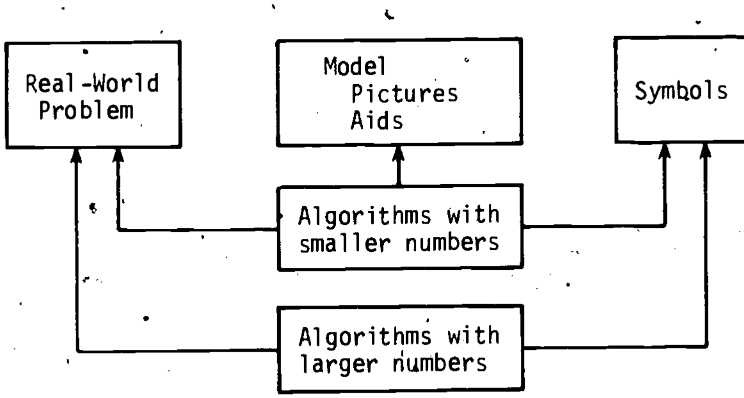
For example: 23 x 56 could be pictured as follows.



Illustrations such as the last one above are not particularly helpful. Children often find that making up a problem is more helpful. Such a problem might be

"There are 23 boxes with 56 books in each.
How many books in all?"

Such a problem places the example in an understandable perspective for the child. Thus, as the numbers become larger it is important to help the child bridge the gap from the real-world problem to symbols.



DIRECTIONS:

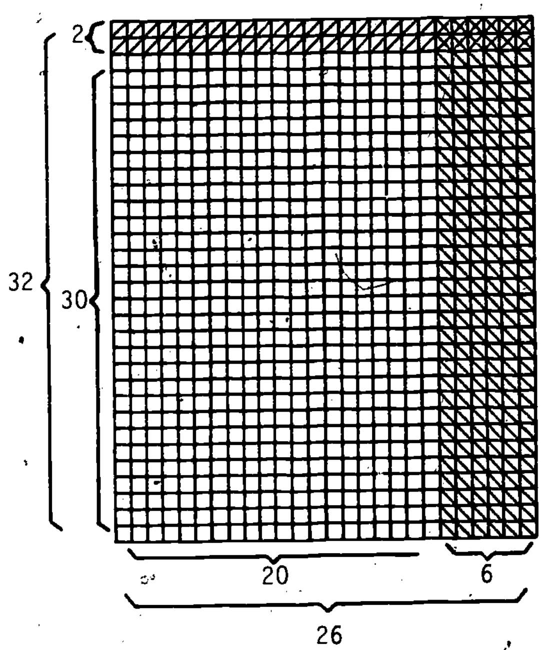
1. Work in pairs. Take turns using Dienes blocks and bundling sticks to show your partner how the following examples could be demonstrated to children. Be sure to relate the aids to the symbols as illustrated in the discussion. In particular, ask each other questions like (a), (b), and (c) on page 70.

   a) 4 x 13   b) 6 x 14   c) 5 x 19

   d) 13 x 34   e) 5 x 120   f) 12 x 21

   How are examples (d), (e), and (f) different from the others? How do they differ from each other? Is it useful in all examples to use the bundling sticks or Dienes blocks?

2. Study the graph below. (work in pairs or individually.)



The graph represents 32 x 26. Using the distributive property you know that:

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$$
\begin{aligned}
32 \times 26 &= 32 \times (20 + 6) \\
&= (32 \times 20) + (32 \times 6) \\
&= (30 + 2) \times 20 + (30 + 2) \times 6 \\
&= (30 \times 20) + (2 \times 20) + (30 \times 6) + (2 \times 6)
\end{aligned}
$$

Color <u>red</u> the part that represents (30 x 20).
Color <u>blue</u> the part that represents (2 x 20).
Color <u>black</u> the part that represents (30 x 6).
Color <u>green</u> the part that represents (2 x 6).

(Use shading if you do not have colored pencils or crayons.)

3. How does the graph in (2) relate to the algorithms A and B shown below?

A

$$
\begin{array}{r}
32 \\
\times\ \ 26 \\
\hline
\left.\begin{array}{r} 12 \\ 180 \end{array}\right\} \\
\left.\begin{array}{r} 40 \\ 600 \end{array}\right\} \\
\hline
832
\end{array}
$$

B

$$
\begin{array}{r}
32 \\
\times\ \ 26 \\
\hline
192 \\
640 \\
\hline
832
\end{array}
$$

a) In A, 12, 180, 40 and 600 are called partial products. What color are these partial products in the grid?

b) In B, 192 and 640 are the partial products. What colors represent these products in the grid?

c) How does this algorithm relate to the mathematical sentences of (2)?

4. The example below shows the standard algorithm for multiplication.

$$
\begin{array}{r}
32 \\
\times\ \ 26 \\
\hline
192 \\
64 \\
\hline
832
\end{array}
$$

How does this algorithm compare to the ones shown in (3)?

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5. Use graph paper to picture any two examples--one from Group A and one from Group B. (Choose two which are different from your partner's.) Then show them to your partner to check and explain each other's work.

Group A	48 x 37	31 x 45	50 x 61	33 x 70
Group B	21 x 235	33 x 127	41 x 212	13 x 311

6. Some transitional algorithms are written below to express 6 x 14.

Transitional		Standard
10 + 4 x 6 60 + ___ = ___	14 x 6 ___ (6 x 4) ___ (6 x 10) ___ Total	14 x 6 ___

a) They are called transitional since they are not the most efficient algorithms, but they can be helpful in building understanding of the meaning of the standard algorithm and in helping children to use it more insightfully. Complete the algorithms and discuss what merits you feel each has. Discuss how these transitional algorithms relate to the work you did with the aids and graph paper earlier.

b) Look back over the work you have done in this activity: using aids, using a grid, using algorithms with partial products and some transitional algorithms. Discuss with your classmates and outline a sequence you might use in presenting simple multiplication algorithms to young children.

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7. Some researchers and educators feel the best way to introduce algorithms to children is to present a problem such as the one which follows and challenge the children to solve the problem any way they can. Then ask various children to tell how they solved it. Since they have not been taught any algorithms they are in effect making up their own.

"Sally counted 37 cartons of eggs
in the dairy case at the supermarket.
How many eggs are in that case?"

Individuals who have tried such a procedure with children have been amazed at the resourcefulness of the children. It seems reasonable that such an experience would foster understanding of algorithmic procedures to be introduced later. What do you think? List some ways that children who have not been taught the multiplicaton algorithms might solve this problem.

8. Discuss in class the role of knowing basic facts throughout the development of the multiplication algorithm.

9. *OPTIONAL: The Cuisenaire rods can be used to model multiplication. Describe or display how the following examples might be modeled using Cuisenaire rods.*

a) 8 x 7 *b) 6 x 12*

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ACTIVITY 14

WRITING AN ACTIVITY CARD OR OUTLINING A LESSON TO ACHIEVE AN OBJECTIVE IN MULTIPLICATION

FOCUS:

Developing lesson plans is a skill which a teacher will find helpful to learn. It is also fun and instructive to compare lessons with other prospective teachers. In this activity you will have a chance to develop and share an outline of a lesson plan.

MATERIALS:

Acetate transparencies, felt-tipped pens.

DIRECTIONS:

1. Read the list of instructional objectives for teaching the multiplication algorithm. In a small group match each of the examples of activities that follow on pages 78-83 with the objectives. Compare your results with other members of your class.
2. Working in groups of two select an objective and prepare a lesson for that objective. (Your instructor may decide to assign the objectives to assure that a wide choice of objectives is represented.) When writing the lesson, you should adopt the following format:

   a) Objective

   b) Materials

   c) Directions

   d) Follow-up exercises.

   Your directions should carefully lead the child to attain the objective. Follow-up exercises should review and extend the concepts developed. The use of cartoons, pictures and any other forms of motivational material is encouraged. <u>Prepare your ac-</u>

tivities as overhead transparencies and be ready to present them at the next class meeting. Your instructor will lead a class discussion at that time.

3. Are there any objectives for which an example is not provided? If so, create one.

A SEQUENCE OF OBJECTIVES
FOR DEVELOPING THE MULTIPLICATION ALGORITHM

1. The child will be able to state the products for all multiplication combinations through 9 x 9.
2. The child will be able to find the product when a multiple of ten is multiplied by a single-digit number.
3. The child will be able to find the product when any two-digit number is multiplied by a single-digit number, using concrete and pictorial aids.
4. The child will be able to find the product when any two-digit number is multiplied by a single-digit number, using paper-and-pencil computations.
5. The child will be able to find the product when any three- (or more) digit number is multiplied by a single-digit number, performing computations with pencil and paper.
6. The child will be able to find the product when any two-digit number is multiplied by a multiple of ten.
7. The child will be able to find the product when any two-digit number is multiplied by a two-digit number, using concrete and pictorial aids.
8. The child will be able to find the product when any two-digit number is multiplied by a two-digit number, using paper-and-pencil computations.
9. The child will be able to find the product when any three- (or more) digit number is multiplied by a two-digit number using paper-and-pencil computations.

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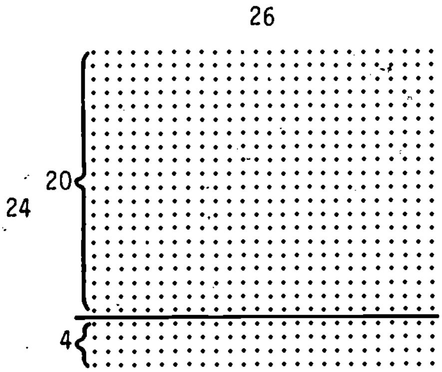
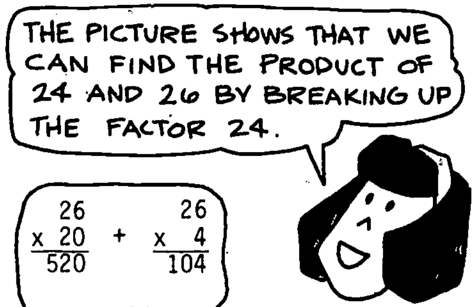
Objective #______



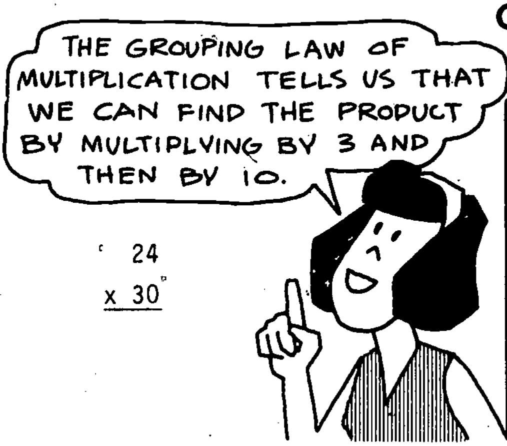


$$\begin{array}{r} 26 \\ \times\ 20 \\ \hline 520 \end{array} + \begin{array}{r} 26 \\ \times\ 4 \\ \hline 104 \end{array}$$

$$= 624$$

A

Objective #______



$$\begin{array}{r} 24 \\ \underline{\times\ 30} \end{array}$$

$$\begin{aligned} 24 \times 30 &= (24 \times 3) \times 10 \\ &= 72 \times 10 \\ &= 720 \end{aligned}$$

B

Objective #______

$$\begin{array}{r} 423 \\ \times\ \ 21 \\ \hline \\ \\ \hline \end{array}$$

(1X423)

(20X423)

C

100

Objective #______

$$
\begin{array}{r} 294 \\ \times\ \ 4 \\ \hline \end{array}
$$

4X4

4X90

4X200

______

D

Objective #______

$$
\begin{array}{r} 43 \\ \times\ \ 2 \\ \hline 86 \end{array} \longrightarrow \begin{array}{r} 43 \\ \times\ 20 \\ \hline \end{array}
$$

E

Objective #______

$$
\begin{array}{r} {\scriptstyle 2} \\ 24 \\ \times\ \ 6 \\ \hline 144 \end{array}
$$

F

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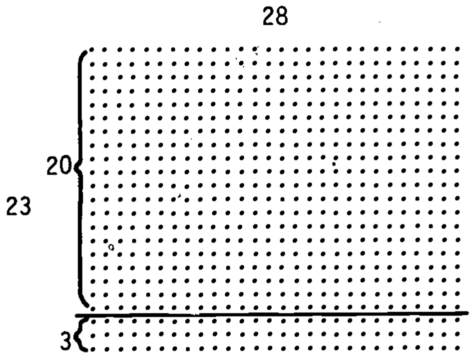
Objective #______




```
   28
 x 23
      (3 x 28)
_____ (20 x 28)
```

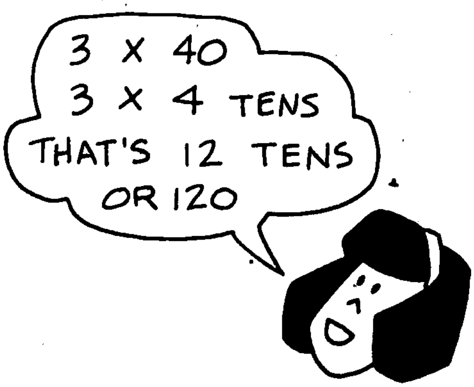
G

Objective #______

```
   43
 X 21
 ----
   43    (1X43)
  860    (20X43)
 ----
  903    (21X43)
```

H

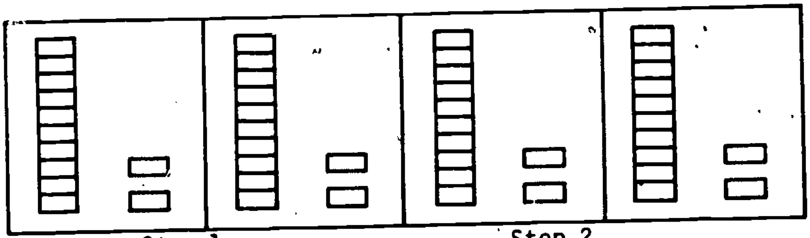
Objective #______



3 x 4 = 12
3 x 40 = _____

I

Objective #_____



Step 1

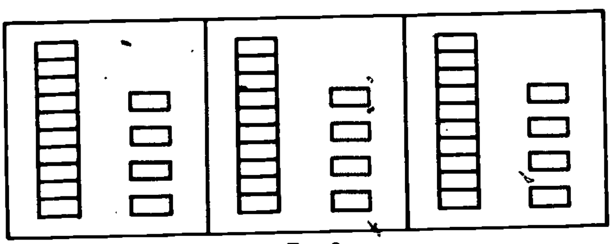
T	O
1	2
X	4
	8

Step 2

T	O
1	2
X	4
	8
4	0
4	8

J

Objective #_____



T	O
1	4
X	3
1	2
3	0
4	2

K

Objective #_____

$$\begin{array}{r} 31 \\ \times\ 3 \\ \hline \end{array}$$

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Objective #______

(A)	3 x 3	(H)	8 x 8
(B)	7 x 8	(I)	4 x 7
(C)	3 x 4	(J)	6 x 8
(D)	6 x 7	(K)	5 x 8
(E)	4 x 5	(L)	6 x 0
(F)	7 x 7	(M)	9 x 9
(G)	3 x 9	(N)	7 x 1

M

Objective #______

$$\begin{array}{r} 368 \\ \times\ 48 \\ \hline \end{array}$$

N

Objective #______

$$\begin{array}{r} {\scriptstyle 2} \\ 217 \\ \times\ \ 3 \\ \hline 651 \end{array}$$

O

Objective #______

208
X 6
______

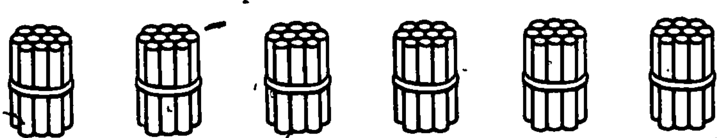
P

Objective #______

48
X 37
______
336
1440
______
1776

Q

Objective #______



How many sets of 10?

Write the numeral for 6 tens.

Solve 6 x 10 = ______

R

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ACTIVITY 15

DISCUSSION OF STUDENT ACTIVITIES

FOCUS:

In this follow-up to Activity 14 you will have a chance to evaluate peer presentations of activities associated with the teaching of the multiplication algorithm and to summarize important pedagogical principles involved in the teaching of the multiplication algorithm.

MATERIALS:

Transparencies prepared in Activity 14, overhead projector.

DIRECTIONS:

1. Selected groups will make presentations of the elementary school activities they prepared in Activity 14.
2. Ask any questions which you have pertaining to the development of the multiplication algorithm.

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ACTIVITY 16

PATTERNS USING THE MULTIPLICATION ALGORITHM

FOCUS:

Children are fascinated with the many interesting relationships among numbers. This activity will present a few patterns in multiplication which children may find intriguing and helpful.

DIRECTIONS:

1. a) The product of 11 and another number suggests one interesting pattern. The problem 11 x 32 is worked on the right. What pattern is suggested?

$$\begin{array}{r} 32 \\ \times\ 11 \\ \hline 32 \\ 320 \\ \hline 352 \end{array}$$

   b) Explore the pattern of products when 111 or 1111 is a factor. (A hand calculator is very useful when exploring patterns in multiplication, particularly when the numbers involved are large.)

2. a) What patterns are suggested by products whose factors are 101 or 99? Try some examples.

   b) Can you write a general rule for factors which are one more or one less than a power of ten?

3. To multiply a number by 25 it may be easier to think of 25 as one-fourth of 100. Thus in multiplying 14 x 25 one can think of dividing 14 by 4 and multiplying by 100. In less formal terms, think "How many 4's in 14?" (3 and 2 left over) "Then 14 x 25 is 300 plus two 25's or 50. The total is 350." Try 17 x 25, 24 x 25, etc.

4. A pattern useful in aiding the development of the multiplication algorithm is illustrated below.

	H	T	O
$2 \times 43 =$		8	6
so			
$20 \times 43 =$	8	6	0

a) State this pattern in general terms.

b) How is the associative property used in this pattern?

5. Explain the relationship between the following pattern and the algebraic manipulation learned in grade 9.

$$\begin{aligned} 31 \times 29 &= (30 + 1)(30 - 1) \\ &= 30^2 - 1 \\ &= 900 - 1 = 899 \end{aligned}$$

$$\boxed{(a + b)(a - b) = a^2 - b^2}$$

6. Find another multiplication pattern, record it, and share it with your classmates. Make a list of the patterns you collect for future use.

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ACTIVITY 17

USING ESTIMATION IN SOLVING MULTIPLICATION PROBLEMS

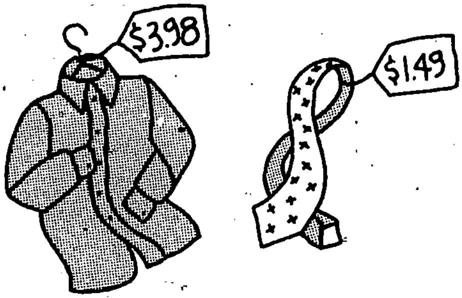
FOCUS:

In this activity the role of estimating in solving some multiplication problems will be explored.

DISCUSSION:

Algorithms are generalized procedures for computing exact answers. Often in everyday situations less formal computational procedures for finding approximate answers are required. These informal procedures are as important for children to learn as the more formal paper-and-pencil procedures. The informal procedures often take the form of mental rounding of numbers and simple calculations. Skill in such mental computation, just as in more formal computation must be based on sound mathematical concepts.

For example, if one wanted to calculate the cost of three boxes of cereal at 48¢ a box one could think: 48¢ is near 50¢ and 3 x $.50 is $1.50. Many problem situations require an estimate of the answer or simply a knowledge of whether the answer is greater or less than a given amount. Many problems like the one pictured below require an estimate.



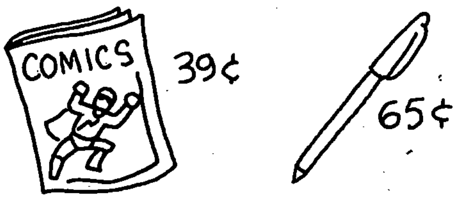
IS $5.00 ENOUGH FOR BOTH?

This activity will present you with some problems for which various strategies could be used in the solution. Since there are numerous strategies you should compare your thinking with that of classmates.

DIRECTIONS:

1. Three problems similar to those found in elementary school textbooks are presented below. Under each problem are parts of <u>two</u> strategies for solving it. For each problem finish the strategies and then write another problem that might be solved using the same strategies.

   a) Jim has been given \$1 to buy a comic book and a pen. Will one dollar be enough? How do you know?

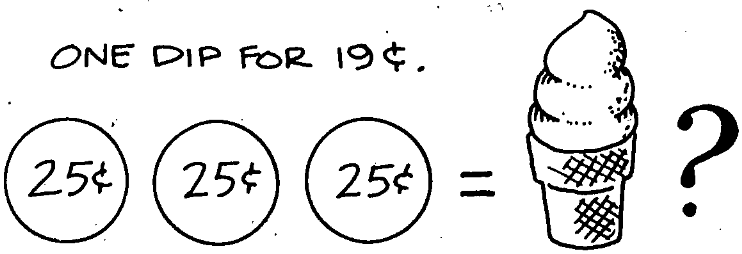


65 AND 35 ADD UP TO A DOLLAR. SO...

60 AND 30 IS 90. 5 AND 9 IS MORE THAN 10. SO...

   b) Jane has three quarters. Ice cream cones cost 19¢ a dip. Does Jane have enough for three dips?

ONE DIP FOR 19¢.



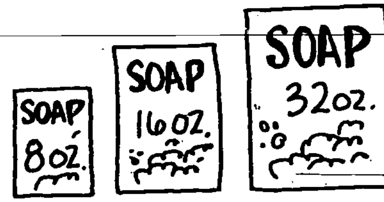
19 IS LESS THAN 25. THREE QUARTERS. THREE DIPS. SO...

19 IS ABOUT 20. THREE 20'S ARE 60. SO...

c) Soap

8 oz. ... 40¢
16 oz. ... 80¢
32 oz. ... 150¢

Which size is the best buy?



8-OUNCE BOX
8 FIVES MAKE 40

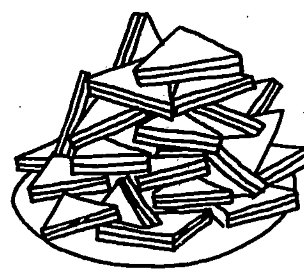
16-OUNCE BOX
16 FIVES MAKE 80

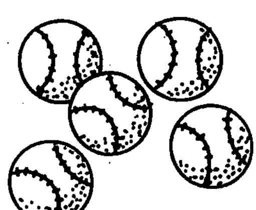
32-OUNCE BOX
32 FIVES ...

TWO 8'S MAKE 16 AND
TWO 40'S MAKE 80.
TWO 16'S MAKE 32 AND...

2. Suggest some strategies which may be used to solve the following problems.

22 guests are coming to a party. If each guest is expected to eat three sandwiches, are 60 sandwiches enough?





Baseballs cost $1.98 (including sales tax). I have $10. Can I buy as many as five?

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3. Often exact answers are computed mentally. Describe two ways a person might compute the answer to the following problem without using the usual algorithm.

4. Briefly summarize the thinking strategies that were suggested or that you used to solve the problems in this activity.

5. Suggest some ways in which children can be encouraged to solve problems using mental estimation.

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ACTIVITY 18

NONSTANDARD ALGORITHMS FOR MULTIPLICATION

FOCUS:

The history of algorithm development includes many interesting procedures. You will become acquainted with various multiplication algorithms, some of which date back several centuries.

DIRECTIONS:

Five examples of multiplication algorithms are presented below. Read each of these and answer the questions which pertain to each algorithm. Children love to learn these "strange" algorithms. It seems reasonable that these algorithms help improve children's understanding and appreciation of multiplication. It is hoped that you too will enjoy them!

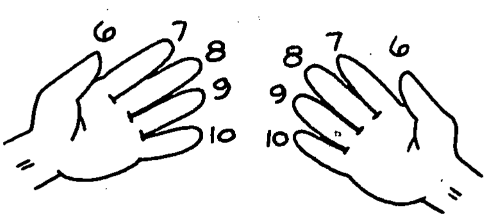
I. Finger Multiplying

This method is said to have originated in Roman times and is still a fairly common practice among Rumanian peasants.

A. How it works:

The method is used for basic combinations involving factors between 5 and 10 and assumes a knowledge of multiplication facts up to 5 x 5. To multiply 9 x 7:

1. Hold your hands in front of you with the palms toward you. Then number the fingers on each hand from 6 - 10 beginning with the thumbs.



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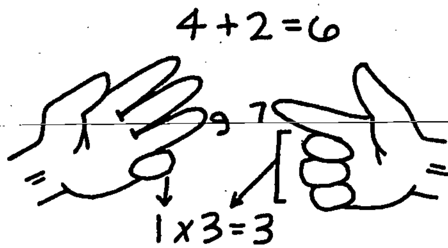
2. To find the product 9×7 touch the 9 finger of one hand to the 7 finger of the other. Bend or flex the fingers <u>below</u> the two that are touching on each hand.



3. The sum of the numbers of fingers extended is the tens digit of the product. (In this case, $4 + 2 = 6$.)

4. The product of the numbers of fingers closed (flexed) on each hand is the units digit. (In this case 1×3.)

5. Hence, $9 \times 7 = 63$.

B. Work the following examples using the finger method.

$$\begin{array}{r} 9 \\ \times\ 8 \\ \hline \end{array} \qquad \begin{array}{r} 8 \\ \times\ 8 \\ \hline \end{array} \qquad \begin{array}{r} 9 \\ \times\ 6 \\ \hline \end{array}$$

C. Try to determine a mathematical explanation of why this algorithm works.

<u>Hint:</u> For example, $9 \times 7 = (10 - 1) \times (10 - 3)$

[where 1 and 3 are the number of fingers bent on each hand]

$= 100 - (1 \times 10) - (3 \times 10) + (1 \times 3)$

$= 100 - 10(1 + 3) + (1 \times 3)$

$= 10[10 - (1 + 3)] + (1 \times 3)$

[where $10 - (1 + 3)$ is the number of extended fingers]

$= (10 \times 6) + 3$

$= 63$

Now generalize by letting a and b be the number of fingers bent on each hand, in which case the numbers to be multiplied are $(10 - a)$ and $(10 - b)$.

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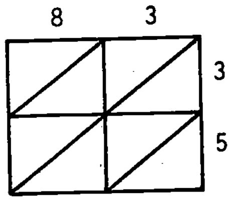
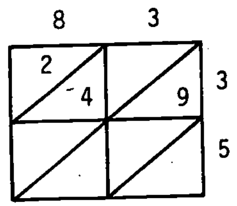
II. Lattice Method

In 1478 a book printed in Treviso, Italy, showed a method of multiplying called the Gelasia (lattice) method.

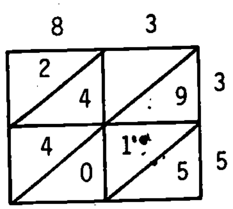
A. How it works:



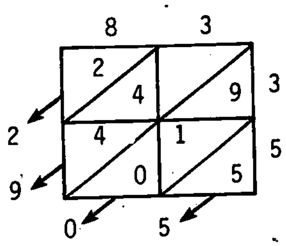
1. Original problem: 83 x 35. One factor is written horizontally and the other written vertically. Diagonals are drawn for each square as indicated.



2. Multiply 3 x 3 = 9
Multiply 3 x 8 = 24
(Notice how the two-digit numeral is placed.)



3. Multiply 5 x 3 = 15
Multiply 5 x 8 = 40



4. Beginning at the lower right-hand corner and moving to the left, add the numbers in each diagonal channel and record the sums as shown, carrying where necessary.

e.g. 5
0 + 1 + 9
4 + 4
2

The product (reading left to right) is 2905.

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B. Work the following examples using the lattice method.

48	70	267
x 27	x 44	x 48

C. Develop a mathematical explanation of why the algorithm works.

III. Napier's Bones

Late in the sixteenth century a Scotsman, John Napier, improved on the lattice method by inventing strips (originally made of bones or ivory and therefore known as Napier's Bones) on which products from a multiplication array were shown. A portion of Napier's array is shown below. Each column is the first ten multiples of the number shown at the head of that column.

A. How it works:

To multiply 374 x 89:

	3	7	4
1	/3	/7	/4
2	/6	1/4	/8
3	/9	2/1	1/2
4	1/2	2/8	1/6
5	1/5	3/5	2/0
6	1/8	4/2	2/4
7	2/1	4/9	2/8
8	2/4	5/6	3/2
9	2/7	6/3	3/6
0	0	0	0

1. Take the columns headed by 3, 7, and 4 and place them as shown.

2. To find 9 x 374, consider the row headed by 9:

9	2/7	6/3	3/6
3	3	6	6

Add diagonal channels from right to left: 6, 3 + 3, 7 + 6, 2, carrying as you go: 3 3 6 6.

3. Follow the same procedure as in 2) for finding 8 x 374. Adding diagonal channels we get: 2 9 9 2. Hence 80 x 374 = 29920.

4. Add:

$$\begin{array}{r} 3366 \\ +\ 29920 \\ \hline 33286 \end{array}$$

374 x 89 = 33286

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B. Work the following examples using the Napier Method.

$$\begin{array}{r} 74 \\ \times\ 18 \\ \hline \end{array} \qquad\qquad \begin{array}{r} 327 \\ \times\ 107 \\ \hline \end{array}$$

C. Try to determine a mathematical explanation of why this algorithm works.

IV. Doubling

To multiply the numbers 292 and 43 using the doubling method, write the numbers 1 and 292. Progressively double each of these numbers, as shown below, until a subset of the numbers in the left-hand column sums to 43. Cross out those numbers not used in the sum as well as the corresponding numbers in the right-hand column. The sum of the numbers in the right-hand column, which are not crossed out, is the product of 292 and 43.*

1	292
2	584
~~4~~	~~1168~~
8	2336
~~16~~	~~4672~~
32	9344
43	12556

Try these examples using this technique.

$$\begin{array}{r} 722 \\ \times\ 128 \\ \hline \end{array} \qquad\qquad \begin{array}{r} 1128 \\ \times\ \ 89 \\ \hline \end{array}$$

*A hint in explaining how this works is to write the number 43 in base two and note the positions of the ones and zeros.

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V. Russian Method

This method is also referred to as the "doubling method." Actually it is a rather simple technique where the only skill required is multiplication and division by 2.

A. How it works:

To multiply 42 by 294

~~42~~	~~294~~	Original numbers
21	588	Multiply 294 by 2, divide 42 by 2 and ignore the remainder.
~~10~~	~~1176~~	Multiply 588 by 2, divide 21 by 2 and ignore the remainder.
5	2352	
~~2~~	~~4704~~	Continue this process until division by 2 is no longer possible.
1	9408	
	12348	Cross out both numerals in each row for which the halved one is even (42, 10, 2). Add the remaining numerals on the right. The result is the product (12,348).

B. Work the following examples using the Russian Method.

$$\begin{array}{r} 29 \\ \underline{\times\ 40} \end{array} \qquad\qquad \begin{array}{r} 264 \\ \underline{\times\ \ 53} \end{array}$$

C. Try to determine a mathematical explanation of why the algorithm works.

Hint: If 42 is written in base two it is 101010 or
$(1 \times 32) + (0 \times 16) + (1 \times 8) + (0 \times 4) + (1 \times 2) + (0 \times 1)_{two}$.
Note that $32 \times 294 = 9408$; $8 \times 294 = 2352$; $2 \times 294 = 588$; and that $32 + 8 + 2 = 42$.

ACTIVITY 19

INTRODUCING THE DIVISION ALGORITHM

FOCUS:

In this activity you will have an opportunity to consider some techniques for introducing the division algorithm. In particular, the relationship between real-world problems and the related computations will be emphasized.

MATERIALS:

Dienes Blocks and bundling sticks.

DISCUSSION:

Many elementary teachers claim that division is the most difficult mathematical skill to teach in the elementary school. Whether it is the most difficult is a matter of speculation but the fact that many teachers feel that way should alert you to the fact that special attention must be given to provide a careful development. As with multiplication, introduction to the division algorithms should be accompanied with many aids. The use of manipulatives such as Dienes blocks or bundling sticks seems to be a necessary step in the child's understanding. Again, as in multiplication, as the numbers get larger one must rely on going directly from the real-world problem to the symbols since concrete representations of large numbers would be confusing.

An example of a way to introduce division is briefly summarized here. Note the illustrated progression from real world ⟶ model ⟶ symbols. Also note the computational form used in the example on the following page.

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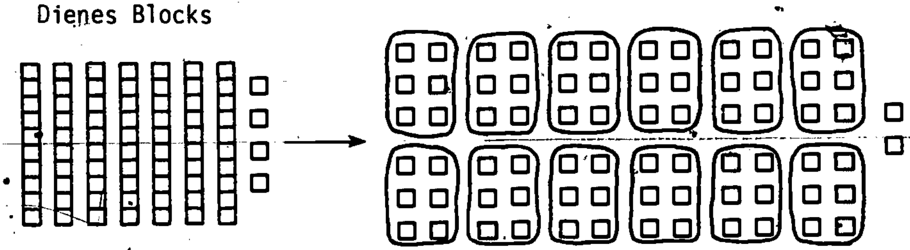
EXAMPLE A

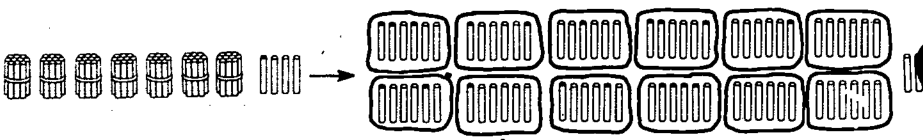
"Mother made a batch of 74 cupcakes. She wants to freeze them in packages of 6 each. How many packages will she freeze?"

Dienes Blocks



trading longs for units

Bundling Sticks



unbundled sticks

CUPCAKES PER PACKAGE

TOTAL NUMBER OF CUPCAKES

```
6)74
  60    10 ← TEN PACKAGES OF 6 CONTAIN 60 CUPCAKES, LEAVING 14
  14
  12     2 ← TWO PACKAGES OF 6 CONTAIN 12, LEAVING 2
   2    12
```

TWELVE PACKAGES WITH 2 CUPCAKES LEFT OVER; ALL 74 CUPCAKES ACCOUNTED FOR

Note that in division of whole numbers there are always two "answers." One answer tells how many equivalent sets of a given number can be partitioned from a given number; the other tells how many are left. (When we say "there is no remainder," we mean that the remainder is zero.)

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Study Example B below and answer the questions about the numbers used in the computation. (Some of the questions have been answered for you.)

EXAMPLE B

"Sally collected 119 seashells at the seashore. She wished to display 9 shells on each piece of cardboard. How many pieces of cardboard did she fill? How many shells did she have left?"

$$
\begin{array}{r|l}
9\,\overline{)\,119} & \\
\underline{90} & 10 \\
29 & \\
\underline{27} & \underline{3} \\
2 & 13\ \text{R}2
\end{array}
$$

Referring to the story and computation above, answer the following questions.

a) What does 9 represent? (The number of shells to be put on each piece of cardboard.)

b) What does 119 represent?

c) What does 10 represent? (Pieces of cardboard with 9 shells on each.)

d) What does 90 represent?

e) What does 29 represent? (Shells remaining after 10 pieces of cardboard are filled.)

f) What does 27 represent?

g) What does 3 represent?

h) What does 13 represent?

i) What does 2 represent?

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DIRECTIONS:

1. In groups of three or four use Dienes blocks and/or bundling sticks to illustrate the following real-world problems. Do the computation for each problem, and answer questions like (a) - (i) in Example B for each problem. That is, tell what <u>each</u> number in the computation means.

   a) Iris had a book with 98 pages to read in one week. How many pages should she read each day if she wanted to read the same number of pages each day and finish the book in a week?

   b) Jack was putting pictures in his album. If he had 107 pictures and wished to put eight pictures on a page, how many pages would he fill? How many pictures would he have left?

   c) If a carton holds six cokes, how many cartons are needed to package 218 cokes? How many bottles are left over?

2. For many of you the computational form displayed in Example B may be unfamiliar. This form is generally referred to as the "scaffold" form. Often it is used to introduce division since it has the advantage of writing the numerals as children have previously seen them. Compare, for example, the numbers used in the scaffold form with those in the so-called standard form.

Scaffold	Standard
9)119 90 | 10 29 27 | 3 2 13 R2	13 R2 9)119 9 29 27 2

   a) What number does the 9 right under 119 in the standard form illustrated above really represent?

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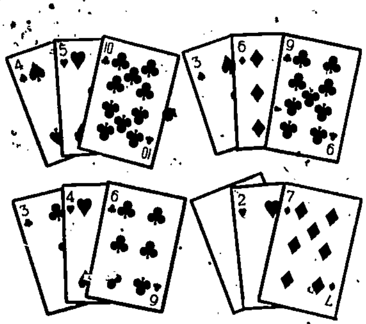
b) We might say, after subtracting, that we "bring down" the 9 from 119. What is actually occurring?

3. Consider and take a position on the following question: "Should the scaffold form be used to introduce division to young children?" Discuss your position with members of your group. Summarize the pros and cons raised in the discussion.

TEACHER TEASER

The cards below in the first three sets are displayed according to a definite system. Find the underlying principle and supply the missing card in the fourth set.



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ACTIVITY 20

THE SCAFFOLD FORM VS. THE STANDARD FORM OF THE DIVISION ALGORITHM

FOCUS:

In this activity you will have a chance to consider the problems related to the scaffold and standard algorithms.

DISCUSSION:

In Activity 19 you were introduced to the scaffold form of division computation. The scaffold form is generally recognized as a useful way of introducing division since it builds upon children's familiarity with the numerals used and serves to relate the real-world problem and the computation. This activity will provide an opportunity for you to practice using the scaffold and standard form and to make some judgments about each.

The scaffold form is sometimes called the "stacking" form when the numbers are placed on top. The scaffold or stacking algorithms are often referred to as transitional algorithms. Some examples appear on page 103.

DIRECTIONS:

1. Study the examples of the scaffold and stacking algorithms given on page 103. In what way is John's work like Patty's? In what way is David's work like Jean's? Is there any advantage to going through a stage like Jean's and David's before working the problems as John and Patty do?

2. The set of division examples given below represents a progression which might be found in the elementary school from grade 3 through grade 6. Note that the last few examples include division with decimals. Work the division examples using both the scaffold (or stacking) form <u>and</u> the standard form. There are two questions which you should be prepared to discuss and answer after working these examples. These questions are:

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Examples of Transitional Algorithms		Standard Algorithm
Scaffold	Stacking	
John 6)2237 1800 | 300 437 420 | 70 17 12 | 2 5 | 372 R5	Jean 372 R5 2 10 30 30 100 200 6)2237 1200 1037 600 437 180 257 180 77 60 17 12 5	372 R5 6)2237 18 43 42 17 12 5
David 6)2237 1200 | 200 1037 600 | 100 437 300 | 50 137 120 | 20 17 12 | 2 5 | 372 R5	Patty 372 R5 2 70 300 6)2237 1800 437 420 17 12 5	

Which form seems easier to introduce division to children?

Which form seems easier to use for examples with larger numbers?

a) $8\overline{)131}$	b) $80\overline{)456}$	c) $30\overline{)1235}$
d) $24\overline{)685}$	e) $35\overline{)2468}$	f) $62\overline{)12578}$
g) $214\overline{)4687}$	h) $1.2\overline{)61.68}$	i) $0.037\overline{)4.699}$

3. Discuss and summarize answers to the two questions asked in (2) above.

4. Use the standard form to explain to a small group of classmates how to do any division example from (2) above. (Role-play, taking turns acting as teacher and as, say, fourth-grade children.) Identify difficulties that you observe and record them after discussion with your instructor.

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ACTIVITY 21

INSTRUCTIONAL SEQUENCES FOR THE INTRODUCTION OF THE DIVISION ALGORITHMS

FOCUS:

In this activity you will have an opportunity to study an instructional sequence used to introduce division.

MATERIALS:

Elementary school mathematics textbooks (grades 3 and 4).

DIRECTIONS:

1. On page 106 is a sequence of hypothetical questions and answers that might be used in introducing the division algorithm.* After studying this sequence answer the questions which follow.
   a) Do you think this sequence of questions and answers might be used as a transition between the scaffold form and the standard form? Why or why not?
   b) Can this sequence be used for an example such as $12\overline{)468}$? Try it with classmates and discuss.
2. Study a textbook series (or other curriculum materials) and briefly outline a sequence of steps used to introduce division up to an example such as $12\overline{)468}$. In your outline include examples which illustrate the sequence.

*Richard O. Kratzer, "A Comparison of Initially Teaching Division Employing the Distributive and Greenwood Algorithm with the Aid of a Manipulative Material" (Ph.D. dissertation, New York University, 1971).

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EXAMPLE: $4\overline{)68}$

Q.: We are going to share 68 among 4. How many tens are there to share?

A.: Six.

Q.: How many tens should each get?

A.: One.

Q.: How many tens are used so far?

A.: Four.

The written work should appear on the chalkboard as follows.

$$\begin{array}{lll} & 1 & \\ \cline{2-3} 4) & 6\text{ tens} & 8\text{ ones} \\ & 4 & \\ \cline{2-3} \end{array}$$

Q.: How many tens are left?

A.: Two.

Q.: Can we share any more tens?

A.: No.

Written work:

$$\begin{array}{lll} & 1 & \\ \cline{2-3} 4) & 6\text{ tens} & 8\text{ ones} \\ & 4 & \\ \cline{2-3} & 2 & \end{array}$$

Q.: What should we do with the two tens which remain?

A.: Trade them in for 20 ones.

Q.: How many ones will there be all together?

A.: 28.

Written work:

$$\begin{array}{lll} & 1 & \\ \cline{2-3} 4) & 6\text{ tens} & 8\text{ ones} \\ & 4 & \\ \cline{2-3} & 2 & 8\text{ ones} \end{array}$$

Q.: How many ones should each get?

A.: Seven.

Q.: Where do we place the answer?

A.: Above the ones place because we are sharing ones.

Q.: How many ones were used?

A.: 28.

Q.: How many ones are left?

A.: Zero (none).

Written work:

$$\begin{array}{lll} & 1 & 7 \\ \cline{2-3} 4) & 6\text{ tens} & 8\text{ ones} \\ & 4 & \\ \cline{2-3} & 2 & 8 \\ & 2 & 8 \\ \cline{2-3} & & 0 \end{array}$$

Q.: What does the answer tell us?

A.: When 68 things are shared among 4, each gets 17 and the remainder is 0.

ACTIVITY 22

DEVELOPING THE DIVISION ALGORITHM FOR LARGER NUMBERS

FOCUS:

In this activity you will have an opportunity to study a sequence of steps which are designed to help children learn to divide with larger numbers. You will also develop a lesson designed to teach a step in this sequence.

MATERIALS:

Elementary mathematics textbook series (grades 3 - 6) and other curriculum materials as desired; acetate transparencies and felt-tipped pens (optional); the Mathematics-Methods Program slide-tape presentation "Developing the Division Algorithm" (optional).

DISCUSSION:

Division is a difficult skill to teach. While you may not have reached this conclusion explicitly you probably suspect that as the numbers get larger, the division algorithm becomes more difficult--and teachers come to depend on children's earlier learnings. Children must become more confident in their ability to manipulate symbols since references to aids or real-world models become quite complicated with larger numbers. As you read "Developing the Division Algorithm," study this sequence carefully. A complete understanding on the part of a teacher is essential if (s)he is to help the child understand.

DIRECTIONS:

Read and study the essay "Developing the Division Algorithm for Larger Numbers" (or view and discuss the slide-tape). After you have finished, choose one of the concepts and develop a lesson to teach it to children. Feel free to consult textbooks or other curriculum materials. The lesson format should be agreed upon with your instruc-

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tor. (S)he may choose to include some of the following aspects:

- list of prerequisite skills
- introduction (motivation, etc.)
- materials to be used
- procedure (including key questions)
- assignments
- evaluation.

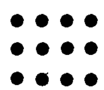
Your instructor may ask you to put your lessons on transparencies, to role-play and teach it to classmates, or to teach it to some children.

"DEVELOPING THE DIVISION ALGORITHM FOR LARGER NUMBERS"

This essay is written in numbered "steps" ((1), (2), etc.) to make it easier for you to refer to in discussion. Not every possible step has been included. Rather, a general overview of steps has been included starting with very early activities and progressing through the division algorithm with large numbers. There will undoubtedly be questions or lack of understanding on some of the steps. Discuss these steps with classmates and/or record them for the seminar in Activity 25.

1. Early work with division begins with extensive readiness activities. In one of these experiences, the child relates division to multiplication. The array is used to represent both a multiplication and division example.

3 rows of 4

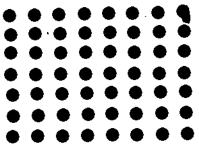


$3 \times 4 = 12$

How many 4's in 12?
$12 \div 4 = 3$

2. Using this relationship, the child recognizes division as finding the missing factor. Knowing that 7 x 8 = 56 enables him to find how many 8's in 56. Note how the array helps the child to solve the number sentence.

7 rows of 8



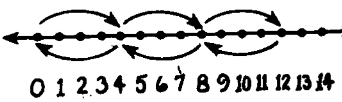
$7 \times 8 = 56$

$56 \div 8 = \square$

3. The relationship between multiplication and division can be further emphasized by asking children to write all possible number sentences for three related numbers (e.g., 3, 9, 27).

$3 \times 9 = 27$	$27 \div 9 = 3$
$9 \times 3 = 27$	$27 \div 3 = 9$

4. Since multiplication can be thought of as repeated addition, one way of presenting division is repeated subtraction. In this case, the number line serves as a good pictorial model for representing division. One question that a teacher might ask here is, "Suppose that a rabbit is at 12; how many hops of 4 does he need to get back home?" (Home is zero.)



$3 \times 4 = 12$ $12 \div 4 = 3$

5. The child has already learned that both addition and subtraction are written in vertical form. In this lesson, the vertical forms for representing multiplication and division are presented. Notice that both operations are presented together using related facts. This continues to show the child the relationship of these two operations.

Sentence	Example
$3 \times 4 = 12$	$\begin{array}{r} 4 \\ \underline{\times \ 3} \\ 12 \end{array}$
$12 \div 4 = 3$	$4\overline{)12}$ with quotient 3

6. The relationship between multiplication and division becomes "strained" when division with remainder is introduced. The thinking pattern of multiplication, however, is still most important. In $29 \div 4$, the child should be helped to think "How many fours in 29? How many left over?" The sentence which expresses this and serves as the basic model for the division algorithm is $29 = \square \times 4 + \triangle$.

$4\overline{)29}$

$29 = \square \times 4 + \triangle$

$7 \times 4 = 28$

There are 7 four's in 29 and 1 left.

$29 = \boxed{7} \times 4 + \overset{\triangle}{1}$

7. Throughout the development of division, the child is encouraged to estimate. This is important when finding the greatest number of 9's less than 58. In practice it may be necessary for the child to make several trials before (s)he finds the correct answer.

How many 9's in 58?

$6 \times 9 < 58$ $\quad$ $7 \times 9 > 58$

$$\begin{array}{r} 6 \\ 9\overline{)58} \\ \underline{54} \\ 4 \end{array}$$

Multiply 6×9 and subtract.

Since $4 < 9$, the remainder is 4.

8. To extend division to include larger numbers, we begin with facts which the child already knows. Since (s)he knows that 2 x 5 = 10 or 10 divided by 5 is 2, ...

Since 2 x 5 = 10

$$\begin{array}{r|l} 2 & \\ 5\overline{)10} & \\ 10 & (2 \times 5) \\ \hline & \end{array}$$

...it is easy to extend this idea to find that 20 x 5 = 100 or 100 divided by 5 is 20.

Since 20 x 5 = 100

$$\begin{array}{r|l} 20 & \\ 5\overline{)100} & \\ 100 & (20 \times 5) \\ \hline & \end{array}$$

9. Estimation continues to play an important role here. The objective is to <u>estimate</u> the greatest number of tens in the quotient rather than to find the exact answer.

To answer the question, "How many 6's in 412?" the child relies on basic facts and the trial procedure.

$50 \times 6 < 412$

$60 \times 6 < 412$

$70 \times 6 > 412$

Best estimate: 6 tens

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10. Children often have difficulty making good estimates. Practice will help, and some teachers allow children initially to make estimates with partial quotients. The procedures all end with the same results, but children should be encouraged to become as efficient as Carol is.

David

5)177	
50	10
127	
50	10
77	
50	10
27	
25	5
2	35 R2

Jean

5)177	
100	20
77	
25	5
52	
50	10
2	35 R2

Carol

5)177	
150	30
27	
25	5
2	35 R2

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11. After dividing two-digit numbers by a single-digit number, the child learns to divide three-digit numbers. To divide 436 by 6 one can begin by asking, "Are there ten 6's?" and then, "What is the greatest number of tens?" Notice how this thinking relies on previous work.

a) After finding the first partial quotient of 70, the process continues by multiplying 6 and 70, subtracting 420 from 436, and then beginning again to decide, "How many 6's are in 16?"

$$\begin{array}{r|l} 6\overline{)436} & \\ \underline{420} & \underline{70} \\ 16 & \end{array}$$

b) When all of the partial quotients are found, the answer is obtained by adding the partial quotients and then recording the total quotient and the remainder. In this problem, the partial quotients 70 and 2 yield a total quotient of 72 and a remainder of 4.

$$\begin{array}{r|l} \multicolumn{1}{r}{72} & \\ 6\overline{)436} & \\ \underline{420} & \underline{70} \\ 16 & \\ \underline{12} & \underline{\ 2} \\ 4 & 72\ \text{R4} \end{array}$$

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12. This entire process relies on the child's ability to estimate and his/her proficiency in multiplication. This same procedure is used for numbers larger than 1000, using the same steps shown above.

7)2568	
2100	300
468	
420	60
48	
42	6
6	366 R6

13. Since division is the inverse operation of multiplication, multiplication can be used to check the results of a division example. In the checking procedure shown here, notice in both the vertical and horizontal forms the position of each part of the division problem.

$436 \div 6 = 72$ R4

$(72 \times 6) + 4 = 436$

$$\begin{array}{r} 72 \\ \times\ 6 \\ \hline 432 \end{array} + 4 = 436$$

14. Throughout the development of the division algorithm, real-world referents are used. In this problem, money is used. Once place value is understood, examples using money are just a simple extension of examples with whole numbers.

$32.89 for gift

7 people share the cost

What is each person's share?

15. After an understanding of the division process has been developed, the "standard" form is introduced. This is done before two-digit divisors are introduced. The questions used in developing the standard form are the same as those used previously. "How many thousand 9's in 7425? (None) How many hundred 9's? How many ten 9's? How many 9's?

```
9)7425 |               825
  7200 | 800        9)7425
   225 |              72
   180 |  20           22
    45 |               18
    45 |   5            45
       | 825 R0         45
                         0
```

16. So far the child has worked only with division examples using one-digit divisors. He has developed the essential skill of estimating the quotient using the basic facts as a necessary tool. Next the child learns to divide using two-digit divisors. (S)He starts with divisors which are multiples of ten.

```
    8                  8
 6)5028            60)5028
   48                 480
    2                  22
```

17. After dividing with multiples of ten the child learns to divide with any two-digit divisor. Rounding such divisors to the nearer ten and estimating are essential steps in this procedure.

```
     8        Think:
 62)5028      62 is near 60
    496       8  60's are 480
     68
```


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18. Subsequent steps in the development of the division algorithm include larger dividends (a) and larger divisors (b). The essential development, however, has been completed at this point. The algorithm is merely expanded to include larger numbers.

(a) $26\overline{)56284}$

(b) $400\overline{)61234}$

$328\overline{)43516}$

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ACTIVITY 23

THE ROLE OF ESTIMATION IN FINDING QUOTIENTS

FOCUS:

It is very helpful for children to have an idea of how large the quotient is prior to processing the division algorithm. This is particularly true when children work at advanced levels with greater numbers, and with short forms of the algorithms. This activity will illustrate the importance of estimation in computing quotients.

DISCUSSION:

In the previous activity it became evident that estimation is a skill that is essential for proficiency in division. Children's ability to estimate seems to depend on the teacher rather than on curriculum materials. Experienced teachers often ask children to estimate answers for any computation before assigning written work, Asking children to estimate answers before paper-and-pencil work takes very little time and the payoff is great. In this activity the estimation is specifically related to the division algorithm.

DIRECTIONS:

1. The questions below pertain to the problems to the right and are representative of the types of questions that are asked children about the quotient. Answer the questions for each problem.

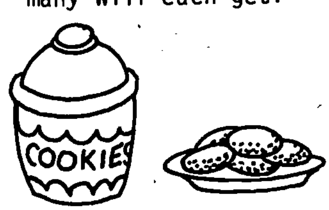
   a) Is there at least one 40 in 290?

      Are there as many as ten 40's?

      How do you know?

      The quotient then is between _____ and _____.

There are 290 cookies for 40 children to share. How many will each get?



b) Does each person carry at least one book?

Does each of the 32 people carry at least 10 books?

Does each carry at least 100 books?

How about 1000 books?

The number of books that each person carries is between _____ and _____.

There are 4125 books to be carried by 32 people. If each carried an equal number, how many does each carry?

c) Is the quotient at least 1?

Is the quotient at least 10?

Is it at least 100?

At least 1000?

The quotient is therefore between _____ and _____.

Solve the following problem:

$$18\overline{)3422}$$

d) In order to encourage children to determine the size of the quotient (the range in which it lies), they might be given a series of examples together with related number sentences. The child is not to solve each of the examples, but is to tell, from the sentences, which quotients are between 1 and 10, 10 and 100, 100 and 1000, and so on.

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$$1 \times 30 = 30$$
$$10 \times 30 = 300$$
$$100 \times 30 = 3000$$

Study the division examples below. Which quotients are between 1 and 10? Which quotients are between 10 and 100?

$30\overline{)121}$ $30\overline{)278}$ $30\overline{)833}$

$30\overline{)356}$ $30\overline{)2861}$ $30\overline{)98}$

e) For the example below, write the approximation sentences that you would use with children, and the questions you would ask.

$$24\overline{)1689}$$

2. In each of the examples in (1), you were able to determine the relative size of the quotient by stating that it was between two numbers.

   a) If, through this process, the child determines that the answer is between 10 and 100, for example, what else does this tell him/her about the answer?

   b) What does it tell him/her about the location of the first digit of the quotient?

3. Suppose that a child worked the following two problems and arrived at the answers shown below.

$$\begin{array}{r} 94 \\ 3\overline{)2712} \end{array} \qquad \begin{array}{r} 69 \\ 7\overline{)4830} \end{array}$$

   a) Why might the child have made these mistakes?

   b) How would an estimation process as outlined in (1) above help to prevent these errors?

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4. Study the two problems below.

$5\overline{)368}$ $5\overline{)621}$

a) How are they alike, and how are they different?

b) How might estimation help in this situation?

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ACTIVITY 24

ERROR DIAGNOSIS AND REMEDIATION IN MULTIPLICATION-DIVISION ALGORITHMS

FOCUS:

Since the algorithms for multiplication and division are more difficult than those for addition and subtraction, you will probably be called upon more often for careful analysis of pupil errors with these topics. A few examples of pupil errors are provided here for you to analyze.

DISCUSSION:

One of the most important tasks a teacher must do is to identify errors and probable causes of the errors and to prescribe a remedial experience (of course, it is even better to teach in such a way as to prevent errors!).

DIRECTIONS:

1. Study the pupil worksheets and find the error pattern shown by each child. To demonstrate that you have found the pattern complete the last two problems on each worksheet as the child would.
2. Give the probable cause(s) of the errors.
3. Outline a procedure which you might use to help the child. For some of the examples direction (2) or (3) has been completed for you.
4. Prepare to discuss your analysis and suggestions for remedial work in the seminar which follows this activity.

WORKSHEET A

Name Jimmy

A. $\begin{array}{r} \overset{1}{3}7 \\ \times 2 \\ \hline 64 \end{array}$ B. $\begin{array}{r} 43 \\ \times 3 \\ \hline 129 \end{array}$ C. $\begin{array}{r} \overset{2}{2}7 \\ \times 4 \\ \hline 88 \end{array}$ D. $\begin{array}{r} \overset{1}{6}4 \\ \times 3 \\ \hline 182 \end{array}$

E. $\begin{array}{r} 26 \\ \times 4 \\ \hline \end{array}$ F. $\begin{array}{r} 42 \\ \times 5 \\ \hline \end{array}$

Probable Cause of Errors: Jimmy knows his multiplication table, or at least it appears that way. He should be congratulated for that! Further, he seems to understand a part of the multiplication algorithm in that he multiplies the ones and tens by the other factor. On the other hand, he does not seem to understand that he must add the tens from the multiplication of the ones in his final answer. For example, in 64 x 3 he forgets to add the 1 ten from the 12 (3 x 4) as he writes 18 tens. He does not understand that 64 x 3 really means

$$\begin{array}{r} 64 \\ \times\ 3 \\ \hline \end{array} \longrightarrow \begin{array}{r} 60 + 4 \\ \times\ 3 \\ \hline 180 + 12 \end{array} = 192.$$

Suggested Remedial Activity:

WORKSHEET B

Name Jane

A.
$$\begin{array}{r} 13 \\ \times 45 \\ \hline 65 \\ 52 \\ \hline 117 \end{array}$$

B.
$$\begin{array}{r} 36 \\ \times 24 \\ \hline 144 \\ 72 \\ \hline 216 \end{array}$$

C.
$$\begin{array}{r} 62 \\ \times 18 \\ \hline 496 \\ 62 \\ \hline 558 \end{array}$$

D.
$$\begin{array}{r} 44 \\ \times 31 \\ \hline 44 \\ 132 \\ \hline 176 \end{array}$$

E.
$$\begin{array}{r} 55 \\ \times 12 \\ \hline \end{array}$$

F.
$$\begin{array}{r} 46 \\ \times 34 \\ \hline \end{array}$$

Probable Cause of Errors:

Suggested Remedial Activity: Jane knows her multiplication facts, but it appears as if she moved too quickly to the standard algorithm. I would try giving her examples such as the following:

1. a) How much is 40 x 13? ________________ (520)

   b) How much is 5 x 13? ________________ (65)

   c) How much is 45 x 13? ________________ (585)

   If the answer to (c) is not the sum of (a) and (b) then I would ask, "What is the difference between 40 + 5 and 45?"

2.
$$\begin{array}{r} 13 \\ 45 \\ \hline \end{array}$$
←(5 x 13)
←(40 x 13)

$$\begin{array}{r} 36 \\ 24 \\ \hline \end{array}$$
←(4 x 36)
←(20 x 36)

In this activity the emphasis is placed upon the meaning of the partial products 40 x ____ and 20 x ____, etc. I would ask Jane where the 40 came from...where the 20 came from.

WORKSHEET C

Name Julie

A.
$$\begin{array}{r} \overset{1}{5}3 \\ \times 86 \\ \hline 318 \\ 414 \\ \hline 4458 \end{array}$$

B.
$$\begin{array}{r} \overset{2}{6}8 \\ \times 43 \\ \hline 204 \\ 262 \\ \hline 2824 \end{array}$$

C.
$$\begin{array}{r} \overset{1}{8}3 \\ \times 26 \\ \hline 498 \\ 176 \\ \hline 2258 \end{array}$$

D.
$$\begin{array}{r} 28 \\ \times\ 3 \\ \hline \end{array}$$

E.
$$\begin{array}{r} 37 \\ \times 52 \\ \hline \end{array}$$

Probable Cause of Errors:

Suggested Remedial Activity:

WORKSHEET D

Name FRED

A. $6\overline{)621}$ = 13 R3

```
  13 R3
6)621
  6
   21
   18
    3
```

B. $4\overline{)833}$ = 28 R1

```
  28 R1
4)833
  8
   33
   32
    1
```

C. $3\overline{)924}$ = 38

```
  38
3)924
  9
   24
   24
```

D. $5\overline{)1043}$

E. $4\overline{)431}$

Probable Cause of Errors: Fred has memorized (more or less) the division algorithm but he has a rather common problem. He doesn't understand what he's doing! (The scaffold transitional form would probably prevent errors like those Fred is making.) In example A he doesn't understand that when he asks himself, "How many 6's in 6" it means "How many 6's in 600?" When he answers "1" he really means 100. Thus, the answer 13 R 3 would not make sense if he were thinking of the meaning. Even if Fred is operating at a pure algorithmic level he should understand that when he "brings down the 2" he should place a 0 in the tens place before he "brings down the 1."

Suggested Remedial Activity:

ERIC

WORKSHEET E

Name: Linda

A. $8\overline{)461}$ = 75 R5

40
61
56
5

B. $6\overline{)357}$ = 95 R3

30
57
54
3

C. $7\overline{)541}$ = 77 R2

49
51
49
2

D. $9\overline{)431}$

E. $6\overline{)825}$

Probable Cause of Errors:

Suggested Remedial Activity:

ACTIVITY 25

SEMINAR

FOCUS:

In this seminar, you will have a chance to clarify any aspects of the teaching of multiplication or division which may be puzzling you.

DISCUSSION:

Both multiplication and division are developed over several years of the elementary school child's life. Since each step is dependent on previous understandings, a primary-grade teacher must know how the instruction (s)he presents will serve as the foundation for later work. Conversely, the upper-grade teacher must be aware of the sequence of learnings the child has experienced in the earlier years so that (s)he can carefully build upon those concepts.

DIRECTIONS:

As a point of departure for the seminar the following questions are given.

1. What is the role of the transitional algorithms in the development of division?
2. Why is it important (or not important) to help the child move from a transitional algorithm to the standard algorithm as quickly as possible?
3. What is the role of the real-world problem ⟶ model ⟶ symbols strategy in developing the division algorithm?
4. Discuss different ways in which estimation might be fostered in the classroom.

REQUIRED MATERIALS

ACTIVITY	MANIPULATIVE AIDS	AUDIO-VISUAL MATERIALS	SUPPLIES AND OTHER
1		Mathematics-Methods Program slide-tape presentation, "Multiplication and Division in the Elementary School," cassette recorder and projector. (Optional)	Elementary school mathematics textbooks, grades 2, 3, and 4.
3			Elementary school mathematics textbooks, grades 3 and 4 (one set per group).
5	17 chips (counters, etc.) per group. (Optional)		
8			Elementary school mathematics textbooks.
11			Heavy paper, markers, scissors; construction paper or 1" graph paper, three dice per group, and a supply of colored counters or chips.

ACTIVITY	MANIPULATIVE AIDS	AUDIO-VISUAL MATERIALS	SUPPLIES AND OTHER
13	Dienes blocks, bundling sticks grouped in tens and ones; Cuisenaire rods (optional).		$\frac{1}{4}$" or 1-cm. graph paper, colored pencils or crayons.
14		Acetate transparencies and felt-tipped pens.	
15		Overhead projector.	
19	Dienes blocks and bundling sticks.		
21			Elementary school mathematics textbooks (grades 3 and 4).
22		Acetate transparencies and felt-tipped pens; the Mathematics-Methods Program slide-tape presentation, "Developing the Division Algorithm," cassette recorder and projector. (Optional)	Elementary school mathematics textbooks (grades 3-6) and other curriculum materials as desired.

Continued from inside front cover

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* * * * * * * * *

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This unit integrates the content and methods components of the mathematical training of prospective elementary school teachers. It focuses on an area of mathematics content and on the methods of teaching that content to children. The format of the unit promotes a small-group, activity approach to learning. The titles of other units are *Numeration, Addition and Subtraction, Rational Numbers with Integers and Reals, Awareness Geometry, Transformational Geometry, Analysis of Shapes, Measurement, Graphs: The Picturing of Information, Number Theory, Probability and Statistics,* and *Experiences in Problem Solving.*

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